DIPLOMA THESIS

Quantization of Supersymmetric Solitons

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Abstract

We investigate the issue of regularization/renormalization in the presence of a non-trivial background in the case of 1+1-(supersymmetric) solitons. In particular we study and compare the commonly employed regularization methods (mode-energy/momentum-cutoff and derivative regularization). We show that the main point for a consistent regularization/renormalization is to find a relation between the "cutoffs" in the vacuum and the nontrivial sector so that they can be related in a consistent manner. For each scheme we give a principle to derive this relation and to perform calculations in a consistent way. These principles are simple and not restricted to two dimensions or supersymmetric theories.

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1 Introduction

1.1 Historical

In August 1834 the technician and marine-engineer John Scott Russell rode along the Edinburgh-Glasgow-channel and watched a boat which was pulled by two horses. Ten years later (1844) he described his observations in a report to the British Association for the Advancement of Science [1]. He wrote that as the boat stopped a large single wave-amplitude with large velocity and constant shape was running down the channel. He persecuted the wave on his horse and lost it after two miles. This was his first encounter with this "singular and beautiful phenomena" which he called "wave of translation". This is the first known (at least to me) mention of what we today call "soliton" or a "solitary wave" and it was forgotten for more than a century. In 1895 Diederik Korteweg and his PhD-student Hendrik de Vries discovered a nonlinear equation, describing water waves, the so called KdV-equation. They showed that their equation posses a solitary wave as solution. But at this time it was seen as an accident that nonlinear equations could be solved explicitly and therefore their discovery was almost forgotten.

In the twentieth century the invention of computers made it possible to investigate nonlinear problems with prospect of success. In 1965, seventy years after its discovery, Norman Zabusky and Martin Kruskal [2] investigated the KdV-equation, which was found to describe different systems. In numerical calculations they discovered that a solitary wave can overtake a slower one and after a complicated, nonlinear interaction the two waves continue moving with unchanged velocity and shape. The residual effect of the interaction is a phase shift in the relative position of the two waves, an effect which is impossible for linear waves. Because of the individual character of these nonlinear waves Zabusky and Kruskal coined the notion *soliton*.

After this work an intensive investigation of nonlinear soliton-bearing equations began and rich connections between different branches of physics and mathematics - scattering theory, lattice dynamic, Kac-Moody-algebras, Verma-modules, cohomology, topology, Potrjagin numbers - were found.

1.2 Current status of research

During the last decade an enormous flurry of activity and also substantial progress has taken place in understanding non-perturbative effects in both supersymmetric field theories and superstring theories [4]. Central to this is the occurrence of extended objects such as solitons and instantons [3], whose masses and actions, respectively, are inversely proportional to coupling constants so that they gain importance in the strongly coupled regime. As first observed in the two-dimensional sine-Gordon theory ([5],[23],[24]), there is the possibility of an intriguing duality between the ordinary elementary quanta of quantum field theory and bound states of solitons.

The consistent regularization and renormalization in the quantization of (supersymmetric) solitons is still an active area of research (see [31],[32],[33] and references therein) with a number of not completely resolved issues. In some special cases certain properties of the quantum theory in the presence of a non-trivial background can be decided without doing explicit calculations of the quantum corrections, such as the saturation of the BPS bound in two-dimensional minimal supersymmetric theories with kink-solitons (SUSY- sine-Gordon and ϕ^4 model) by the use of the residual supersymmetry [34] or the quantum mass of the kink in the case of the minimal SUSY-sine-Gordon model obtained from Yang-Baxter equation assuming factorization of the S-matrix [35],[36]. These methods provide highly non-trivial cross-checks, but depend on special properties of the considered theories and give no general insight into the impact of the presence of a nontrivial background on the renormalization procedure and thus different aspects of the Quantum Field Theory associated to nontrivial classical solutions. These aspects are overruled by "higher" knowledge such as supersymmetry.

In this work we deliberately do not make use of supersymmetry at each step (which does not mean that we neglect or violate it unnecessarily). We mostly carry out the calculations for bosons and fermions separately to point out the different aspects of the influence of the nontrivial background on fermions and bosons. We investigate different regularization schemes (mode-, energy cutoff- and derivative regularization) within the renormalization procedure which are well known in standard perturbation theory and adapt it to the requirements of the presence of a non-trivial background. Besides the resolution of various subtleties we are able to show that the considered regularization schemes, which are very successful and popular in standard perturbation theory, with the necessary modifications are still well working techniques even in the presence of a non-trivial background. The investigations demonstrate that different schemes emphasize different aspects of the nontrivial background for quantum corrections but all of them eventually give the same unambiguous results. We thus are able to solve certain outstanding problems in the computation of the quantum mass of (SUSY) solitons. Over a long period of time different methods gave different answers (see the references in [37]) and there seemed to be no convergence in the results. More recent works [36],[37] cleared up a lot of things in this discussion but also posed new questions, which are still in discussion [31],[32]. Clearly, the resolution of the remaining open points is an important step for reliable further investigations.

The modifications that have to be applied to the different regularization schemes are based on very simple principles which are not restricted to two dimensions or supersymmetric theories. The generality and simplicity of these principles thus pave the way for further investigations in more general cases than 2D-SUSY solitons. Nevertheless minimal 2D-SUSY solitons are still of particular interest. Firstly, the discussion on the consistent renormalization in the presence of exact static classical solutions is concentrated on these models, secondly there exists a

"higher" knowledge due to supersymmetry resp. exact S-matrices which makes it possible to verify these principles "a posteriori" and above all because of their simplicity, so that one can focus on the problem of consistent regularization resp. renormalization in the presence of a static nontrivial background.

The new principles are formulated and used for the calculation of the quantum correction to the soliton mass. When they are respected, all considered methods give the (one loop) mass corrections, now "accepted by all workers in this field" [32], ($m = \sqrt{l}\mu$, where μ is the minimal renormalized mass parameter and $l = 1, 2$ for SG, ϕ^4):

$$
\begin{aligned}
(susy) - SG: \qquad & \Delta M_B = -\frac{\hbar m}{\pi} \\
& \Delta M_F = \frac{\hbar m}{2\pi} \\
& \Delta M_{susy} = \Delta M_F + \Delta M_B = -\frac{\hbar m}{2\pi}
\end{aligned}
$$

$$
\begin{aligned}
(susy) - \phi^4: \qquad & \Delta M_B = \hbar m \left(\frac{1}{4\sqrt{3}} - \frac{3}{2\pi} \right) \\
& \Delta M_F = \hbar m \left(\frac{1}{\pi} - \frac{1}{4\sqrt{3}} \right) \\
& \Delta M_{susy} = \Delta M_F + \Delta M_B = -\frac{\hbar m}{2\pi}
\end{aligned}
$$

Although in the supersymmetric case the bosonic and fermionic corrections ΔM_B, ΔM_F have no physical meaning by themselves we have calculated them separately due to the reasons mentioned above. Only the sum ΔM_{susy} has a physical meaning and that it is the same in both theories is related to its supersymmetric origin.

1.3 Organization of this work and conventions

In section 2 we review some properties of solitons. This section mostly follows reference [3]. In section 3 we first discuss general principles of the quantization of (static) solitons and renormalization. This will be used to calculate the (one loop) quantum corrections to masses of the ϕ^4- and sine-Gordon- kink-soliton solutions in section 4. The main point of this section is a consistent regularization in different "topological" sectors. In section 5 we consider solitons coupled to fermions. Especially the supersymmetric extensions of the bosonic theories of the foregoing sections are considered and the additional (one loop) fermionic quantum corrections to the kink masses are calculated.

Throughout this work we use, except stated otherwise, units so that $c = 1$ and $\hbar \neq 1$, because $\hbar$ will be our main perturbative parameter. The metric signature is $(+, -)$.

2 Classical Solitons

2.1 Introduction

[1]First we consider the massless Klein-Gordon equation in 1+1 dimensions

$$\Box\phi(x,t) = (\frac{1}{c^2}\partial_t^2 - \partial_x^2)\phi(x,t) = 0$$

This equation and its solutions have well known properties

* It is linear and dispersionless.

* Each "well behaved" function of the form $f(x \pm ct)$ is a solution.

* It is a second order equation, and the plane waves $\cos(kx + \omega t)$ and $\sin(k \pm \omega t)$ with $\omega = kc$ form a basis in the space of general solutions.

Thus each "well behaved" function can be expanded according to this basis:

$$f(x - ct) = \int dk[a(k)\cos(kx - \omega t) + b(k)\sin(kx - \omega t)]. \tag{1}$$

For proper functions $a(k)$ and $b(k)$ this is a wave packet traveling in positive x -direction with the velocity c and since all modes have the same velocity the shape of the wave packet is stable, i.e. constant in time, and thus dispersionless.

Because of the linearity of the massless Klein-Gordon equation a linear combination of solutions is again a solution. Thus one can construct a solution built of several wave packets which can travel with different (opposite) velocities. Consider, for instance, two wave packets:

$$f_3(x,t) = f_1(x - ct) + f_2(x + ct)$$

This solution has following properties:

$t \to \infty$: Two widely separated wave packets.

$t = finite$: Collision of the wave packets.

$t \to -\infty$: Again two widely separated wave packets with the same shapes and velocities as before the collision.

Solutions with several wave packets have analogous properties. For the massless Klein-Gordon equation we conclude, that it is *linear and dispersionless* and from this follows:

(i) Shape and velocity conservation of a wave packet.

(ii) Asymptotic shape and velocity conservation after collision of several wave packets.

[1]This is the only section where we use units in which $c \neq 1$.

2.1.1 Klein-Gordon equation in D=1+1

The Klein-Gordon equation,

$$(\Box + m^2c^2)\phi(x,t) = 0,$$

is also linear and a solution basis is again given by plane waves

$$\cos(kx \pm \omega t) \text{ and } \sin(k \pm \omega t). \tag{2}$$

But now for (2) being a solution ω and k must fulfill the following equation:

$$\omega^2 = k^2c^2 + m^2c^4.$$

From this follows that different modes (different k's in (1)) of the wave packet move with different (phase)velocities

$$v(k) = \frac{\omega(k)}{k} = c\sqrt{1 + \frac{m^2c^2}{k^2}}$$

and a wave packet with a certain shape at time $t = 0$,

$$f(x,0) = \int dk[a(k)\cos(kx) + b(k)\sin(kx)],$$

will spread as time moves on. The Klein-Gordon-equation is *dispersive*. Thus solutions of the Klein-Gordon- equation have neither property (i) nor property (ii) of the solutions of the massless Klein-Gordon-equation.

2.1.2 Nonlinear equations

If we neglect the dispersive mass term of the Klein-Gordon-equation and add a nonlinear term instead to get something new, for example

$$\Box\phi(x,t) + \phi^3(x,t) = 0, \tag{3}$$

we also get wave packet solutions which spread with time. For (3) this can be observed by numerical calculations.

In equations with dispersive *and* nonlinear terms, these two effects can balance each other, so that special solutions occur that have the property (i) or even the properties (i) and (ii). Solutions of nonlinear equations with property (i) are called *solitary waves.* Solutions of nonlinear equations with properties (i) and (ii) are called *solitons.* It is common to call both, solitons and solitary waves, solitons (or "lumps" [6]). We will give a more precise definition in the next chapter. These non-dissipative solutions which do not spread out with time form lumps of energy holding themselves together by their own self-interaction.

2.1.3 Solitons and solitary waves

The definition of solitons and solitary waves vary from author to author, but the several definitions are very similar, they differ only for special cases in the characterization of solutions of nonlinear wave equations. We give a definition which is appropriate for our interests and equivalent to other definitions in the cases treated here.

We characterize localized solution by the energy density (Hamiltonian density) of the field configuration (composite function of the fields)

$$\varepsilon(x,t) = F(\phi_i(x,t)).$$

Connected to the energy density is the total energy (Hamiltonian) of the system by

$$E[\phi_i] = \int\limits_{space} dx\varepsilon(x,t)$$

For physical systems the energy is bounded below and we can shift the minimum to zero, i.e. $E_{min} = 0$. With this normalization we define localized solution as follows:

Definition We call solutions of nonlinear wave equations *localized solutions* if the associated energy densities $\varepsilon(x,t)$ have following properties:

$\varepsilon(x,t)$ is localized in space for finite times t, i.e.

(i) $\lim\limits_{x\to\infty} \varepsilon(x,t) \to 0$ fast enough to be integrable

(ii) $\varepsilon(x,t)$ is finite in finite regions of space

For systems with $E[\phi_i] = 0$ *iff* $\phi_i(x,t) \equiv 0$ this definition of localized solutions is equivalent to localized fields, i.e. if one requires $\lim\limits_{x\to\infty} \phi(x,t) = 0 = \lim\limits_{x\to\infty} \partial_\mu \phi(x,t)$.

Definition A *solitary wave* is a non-singular localized solution of nonlinear field equations whose energy density has a space-time dependency of the form

$$\varepsilon(x,t) = \varepsilon(x - ut)$$

where u is an arbitrary velocity(vector).

This means that the energy density moves with constant velocity and constant shape, i.e. undistorted. From this follows that any static localized solution is a solitary wave with $u = 0$. For relativistic or Galilean invariant systems one obtains moving solutions by boosting the static ones. Therefore static nontrivial solutions with localized and finite energy will be of central interest for us.

Definition A solution of nonlinear field equations with N solitary waves with energy densities $\varepsilon_0(x-ut)$ is called a *soliton* if the energy density has following properties:

(i) $$\varepsilon(x,t) \to \sum_{i=1}^{N} \varepsilon_0(x - a_i - u_i t) \text{ as } t \to -\infty$$

(ii) $$\varepsilon(x,t) \to \sum_{i=1}^{N} \varepsilon_0(x - a_i - u_i t + \delta_i) \text{ as } t \to +\infty$$

Where a_i and u_i are the initial positions and velocities and δ_i are constants or constant vectors for higher dimensions.

So solitons are solitary waves whose velocities and shapes of energy densities are asymptotically ($t \to +\infty$) restored to their initial, i.e. pre-collision, ones. The constants (vectors) δ_i represent the displacement of the pre-collision trajectories and should be the sole residual effect of the collision. It is clear that all solitons are solitary waves but not vice versa.

2.2 General properties of scalar solitons in D=1+1

We give a qualitative discussion of possible solutions of nonlinear field equations for scalar fields. We consider the simplest cases, i.e. only one field in $D = 1 + 1$ dimensions. Of special interest are static solutions with localized energy, which are transformed by a boost into moving localized energy-lumps. This fits to the concept of a particle, but with finite extension. The quantum theory of these objects (see part 2) will validate the particle-picture of these extended objects.

The dynamics of such a simple system is described by the Lagrangian (density),

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - U(\phi) = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\phi'^2 - U(\phi) \tag{4}$$

for which we set up following assumptions

(i) $\mathcal{L}$ is 2D-Lorentz-invariant

(ii) $U(\phi)$ is a positive semidefinite function of ϕ and does not depend on the derivatives $\partial\phi$ of the field.

(iii) absolute minima of U are zero, i.e. $U_{min} = 0$

The equations of motion (e.o.m.) are obtained by a variation principle:

$$\delta \int dx^2 \mathcal{L} = 0 \quad \Rightarrow \quad \Box\phi(x,t) + \frac{\partial U}{\partial \phi}(\phi) = 0. \tag{5}$$

In this case the variation of the fields and its derivatives vanish per definition at possible boundaries of the considered space-time interval (for an unbounded space-time $\mathbb{R}^n$ this is always automatically true). This variation principle is consistent with the second order e.o.m. This is not true for first order systems like fermions, where one needs a modified variation principle (see below).

The Hamiltonian (energy) of systems like (4) is given by

$$E[\phi] = \int dx\varepsilon(x,t) = \int dx(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\dot{\phi} - \mathcal{L}) = \int dx[\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\phi'^2 + U(\phi)] \tag{6}$$

where time independent Lagrangian's give time independent Hamiltonians,i.e. $\frac{\partial \mathcal{L}}{\partial \sqcup} = 0 \Rightarrow \frac{dE}{dt} = 0$.

From (6) one can see that the energy $E[\phi]$ is the sum of positive definite terms. Thus to get the minimum of the energy, i.e. the ground state (the vacuum in the quantized theory) of the theory, each term must be minimal. This is achieved by constant fields $\phi(x,t) = g^{(i)}$ for which the potential $U(\phi)$ has an absolute minimum, i.e. $U(g^{(i)}) = U_{min} = 0$ because of our assumptions.

2.2.1 Boundary conditions

The requirement of localized and finite energy solutions implies strong restrictions on the spatial boundary conditions for these fields. For a localized energy density $\varepsilon(x,t)$ i.e. for finite energy $E[\phi]$ the energy density $\varepsilon(x,t)$ must vanish (fast enough) with $|x| \to \infty$ for the considered field-configuration. From (6) one can see that this is only possible if asymptotically $(|x| \to \infty)$ $\partial_\mu \phi \equiv 0$ *and* $U(\phi) \equiv 0$, since all quantities in $\varepsilon(x,t)$ and $E[\phi]$, respectively, are positive.

Let $\phi = g^{(i)}, i = 1...M \geq 1$ be the absolute minima of potential, i.e. $U(\phi = g^{(i)}) = 0$. A necessary condition for $\varepsilon(x,t)$ to vanish asymptotically is that ϕ approaches one of the (constant) minima $g^{(i)}$. Thus the localized energy condition $\varepsilon(x \to \pm\infty) \to 0$ implies for the field the following *solitary wave-boundary conditions*:

$$\lim_{x\to\pm\infty} \phi(x,t) = g^{(i^\pm)} \tag{7}$$

$$\lim_{x\to\pm\infty} \partial_\mu \phi(x,t) = 0 \tag{8}$$

One has to distinguish between the case of one minimum ($M = 1$) and the case of several minima ($M > 1$). If $M = 1$ then for both limits ($x \to +\infty$ and $x \to -\infty$) the field converges to the same value $g^{(i^\pm)} = g$. For $M > 1$ one can have different limits $g^{i^+} \neq g^{i^-}$ or the equal ones, $g^{i^+} = g^{i^-}$.

2.2.2 Static solutions and mechanical analogue

For static solutions the e.o.m. (5) simplify, in our $D = 1 + 1$ case, to an ordinary differential equation:

$$\Box\phi(x,t)_{|static} = -\phi''(x) = -\frac{\partial U}{\partial \phi}(x). \tag{9}$$

This equation is analogous to the Newton equation of motion of a unit mass in a potential $V = -U$ (see fig.1), if one considers ϕ as the coordinate and x as the time, i.e. $\phi(x) \equiv q(t)$. Therefore we discuss the familiar mechanical analogue instead of the original field system and solve the e.o.m. (9) simply by quadrature. This is only possible in $D = 1 + 1$ dimensions . At

Figure 1: The potential $U(\phi)$ (solid) of the field theory and its (one degree of freedom)-mechanical analogue $V(q)$ (dashed) in the case of static fields in $D = 1 + 1$ dimensions. (a) For a field theory with a unique ground state and (b) for degenerated absolute minima of the potential $U(\phi)$.

the end of this section we make some comments on the situation in more general cases. The mechanical analogue has the following properties:

$$\text{Energy}: \quad W = \frac{\dot{q}^2}{2} + V(q) \equiv \frac{1}{2}(\frac{d\phi}{dx})^2 - U(\phi) = constant \tag{10}$$

$$\text{Boundary conditions}: \quad \lim_{x\to\pm\infty} W = 0 \tag{11}$$

$$\text{Action}: \quad S[q] \equiv \int dx[\frac{1}{2}(\frac{d\phi}{dx})^2 + U(\phi)] = E[\phi]_{|static} \tag{12}$$

The boundary condition (11) follows from (7), and (8), (12) follows from (6).

Virial theorem. Integrating (9) with $\int_{-L}^{y} dx\phi'(x)$ gives:

$$\frac{1}{2}\phi'^2 = U \;\Rightarrow\; \phi' = \pm\sqrt{2U(\phi)}. \tag{13}$$

This is the virial theorem for the mechanical analogue and in connection with non-trivial static field configurations it is called the *Bogomol'nyi equation*.

Thus static solutions of the field system are trajectories with finite action, (12), and zero energy, (10) and (11), of the mechanical analogue. We consider these trajectories for two classes of potentials:

a) From fig.1 one can see that for potentials U with a unique minimum there exists no non-singular, nontrivial trajectory with boundary condition (11). A particle starting at the "time" $x = -\infty$ at $\phi_1 = 0$ will never return. The only solution is the trivial one $\phi(x) \equiv \phi_1$, i.e. there exists no static solitary wave (the field is constant in space).

b) In the case of several degenerated minima of U according to (11) the particle must start at one of the minima of U, ϕ_i, and move to one of the neighboring minima $\phi_{i\pm1}$(the field varies in space). It cannot return or go further since all derivatives of ϕ, i.e. "velocities", "accelerations", etc, vanish at the ϕ_i's due to the equation of motion (9) and (13) and boundary the conditions (11):

$$\begin{aligned} U(\phi_i) = 0 \Rightarrow \phi_i{}' &= 0 \\ \frac{\partial U}{\partial \phi}(\phi_i) = 0 = \phi_i{}'' &\quad \text{a.s.o.} \end{aligned}$$

From the mechanical analogue one concludes for the existence of static solitons for theories of the form $\mathcal{L}= \frac{1}{2}(\partial_\mu\phi)^2 - U(\phi)$ (one field):

1. $U(\phi)$ has an unique minimum $\Rightarrow \exists$ one trivial static solution ϕ_1 and $\nexists$ static solitary waves.

2. $U(\phi)$ has n minima $\Rightarrow \exists\ 2(n-1)$ nontrivial static solutions with $\phi(x \to -\infty) = \phi_k$ and $\phi(x \to \infty) = \phi_{k+1}$ or ϕ_{k-1}, and n trivial solutions ϕ_i.

From the mechanical analogue one can also see that the "particle" moves monotonically from one minimum of U (top of the hill) to a neighboring one. Therefore the static solitary wave is a monotonicly increasing or decreasing function. The above considerations are not restricted to a special shape of the potential $U(\phi)$. The main point is the existence of several (at least more than one) degenerated ground states (absolute minima of $U(\phi)$) which can be accompanied by spontaneous breakdown of a symmetry (see below).

Solving by quadrature. As mentioned above, in the simple $D = 1+1$ case static solutions are obtained by quadrature. Integration of the virial theorem gives

$$x - x_0 = \pm \int_{\phi(x_0)}^{\phi(x)} \frac{d\phi}{\sqrt{2U(\phi)}} \tag{14}$$

Because of the boundary conditions the integrand is regular except for $x_0 \to -\infty$ and $x \to \infty$, where x_0 is the integration constant.

Next we consider two special models and investigate further features of solitons (solitary waves) on the basis of them. These two models will also be considered in the quantization procedure (section 3).

2.3 ϕ^4 - theory, the kink

We consider the ϕ^4 - theory in $D = 1+1$ dimensions (not dimensionally reduced) with a mass (quadratic) term which causes "spontaneous symmetry breaking". The ϕ^4-model with the opposite sign of the mass term generates only one unique minimum so that there exist no static solitary solutions, as mentioned above. The Lagrangian, which also fulfills our assumptions (2.2) is given by

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \phi)^2 - U(\phi) \quad , \tag{15}$$

$$U(\phi) = \tfrac{\lambda}{4}(\phi^2 - \tfrac{\mu^2}{\lambda})^2. \tag{16}$$

The potential (16) has the shape of the potential (b) in fig.1. The e.o.m. of this system are

$$\Box\phi(x,t) - \mu^2\phi + \lambda\phi^3 = 0, \tag{17}$$

and for static solutions they read

$$-\phi'' - \mu^2\phi + \lambda\phi^3 = 0. \tag{18}$$

The minima of the potential have the value zero for $\phi_\pm$, where the ground state configurations $\phi \equiv \phi_\pm$ of the system are

$$\frac{dU}{d\phi} = 0 \Rightarrow \phi_\pm = \pm\frac{\mu}{\sqrt{\lambda}}. \tag{19}$$

From our solitary waves- (localized energy) boundary conditions (7) follows that the field must asymptotically approach these values, i.e. $\phi(x \to \pm\infty) \to \phi_\pm$. Thus we have two possible non-trivial static solutions. One evolving with x from ϕ_- to ϕ_+ and a second one in the opposite direction.

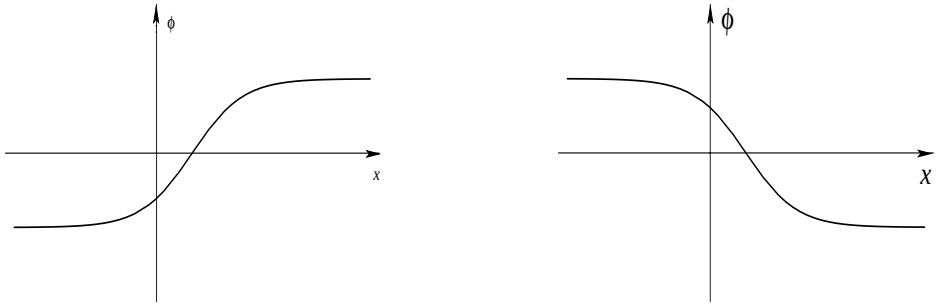


Figure 2: kink and antikink

Integration, static localized solutions With the potential (16) the general integral (14) reads ($\sigma = \pm 1$)

$$x - x_0 = -\sigma \int_{\phi(x_0)}^{\phi(x)} \frac{d\phi}{\sqrt{\lambda/2}(\phi^2 - \mu^2/\lambda)}.$$

This is a standard integral. For fields satisfying $|\ \frac{\sqrt{\lambda}}{\mu}\phi\ | < 1$ and by setting the integration constant $\phi(x_0) = 0$, one obtains by elementary integration and solving for ϕ:

$$\phi_{K_\sigma}(x) = \sigma \frac{\mu}{\sqrt{\lambda}} \tanh[\frac{\mu}{\sqrt{2}}(x - x_0)]. \tag{20}$$

For later considerations we have introduced the sign variable σ. One can easily prove by inserting in (18) that this is a solution. For fields satisfying $|\ \frac{\sqrt{\lambda}}{\mu}\phi\ | > 1$ one gets functions involving coth. These cannot satisfy the boundary conditions. Thus we have two nontrivial static solutions which are called *kink* ($\sigma = +$) and *antikink* ($\sigma = -$), and shown in fig.2. These two solutions can be traced back to the two signs of the Bogomol'nyi equation (13):

$$\phi'_{K_\sigma} = -\sigma\sqrt{2U(\phi_{K_\sigma})}.$$

As one can see, these solutions are singular for $\lambda \to 0$. Thus they cannot be obtained by perturbation theory starting from the linear equations ($\lambda = 0$). Thus the kink ϕ_{K_+} and the antikink ϕ_{K_-} are *non-perturbative* results.

2.3.1 Spontaneous symmetry breaking

We give some comments on the symmetry and its spontaneous breakdown of this model. Spontaneous symmetry breaking will be of importance in the quantization procedure but it is a feature which is already present at the classical level and does not come from quantization. In (quantum) perturbation theory one generally expands the Lagrangian around a fixed (classical) field-configuration ϕ_{cl} (see below). Thus the Lagrangian becomes a function of the perturbations η around the fixed configuration ϕ_{cl}, i.e.[2]

$$\mathcal{L}(\phi) = \mathcal{L}(\phi_{cl} + \eta) = \tilde{\mathcal{L}}(\eta).$$

Now assume that the theory has a linear symmetry, so that the Lagrangian is invariant (up to total divergences) under the linear transformations $T\phi$ of the field, i.e

$$\mathcal{L}(T\phi) = \mathcal{L}(\phi) = \mathcal{L}(\phi_{cl} + \eta) = \tilde{\mathcal{L}}(\eta). \tag{21}$$

[2]All equations resp. inequalities are to be understood modulo total divergences.

On the other hand, if the "ground state" ϕ_{cl} is not invariant under this transformation (if the boundary conditions do not respect the symmetry), i.e $T\phi_{cl} = \bar{\phi}_{cl} \neq \phi_{cl}$, one has

$$\mathcal{L}(T\phi) = \mathcal{L}(T\phi_{cl} + T\eta) = \mathcal{L}(\bar{\phi}_{cl} + T\eta) \neq \tilde{\mathcal{L}}(T\eta) \Longrightarrow \tilde{\mathcal{L}}(T\eta) \neq \tilde{\mathcal{L}}(\eta).$$

This effect is called spontaneous symmetry breaking. Actually it is just hidden symmetry, since by writing the Lagrangian as a function of the perturbations η the symmetry of the system is no longer visible although still present as one can see in (21). The effect of spontaneous symmetry breaking gives rise to a rich structure in quantum field theory and particle physics (Higgs effect, Goldstone theorem). In the quantum theory the "ground states" ϕ_{cl} are usually one of the minima of the potential, i.e. the configuration with the lowest energy, and the associated quantum mechanical state is the vacuum $|0\rangle$. Therefore in quantum theory on calls a symmetry spontaneously broken if the vacuum state is not annihilated by the symmetry transformation, i.e.

$$\mathcal{T}\,|0\rangle \neq 0.$$

A less trivial "ground state" is a non-trivial classical solution like our kinks. Their quantum theory will be the main part of this work.

Let us return to the ϕ^4 model. The Lagrangian (15) and the associated action is symmetric under the parity transformations $x \to Px = -x$ and separately for the $\mathbb{Z}_2$ (gauge) transformation $\phi \to Z\phi = -\phi$. By the $\mathbb{Z}_2$ (gauge) transformation the two minima $\phi_\pm$(19) are not invariant but transformed into each other, i.e. $Z\phi_\pm = \phi_\mp$. This is the "classical" situation of spontaneous symmetry breaking. But from (20) one can see that for the kink and antikink ϕ_{K_σ} both symmetries transformation the two solutions into each other:

$$\begin{aligned} \text{parity}: \quad & \phi_{K_\sigma}(Px) = \phi_{K_\sigma}(-x) = -\phi_{K_\sigma}(x) = \phi_{K_{-\sigma}}(x) \\ \mathbb{Z}_2: \quad & Z\phi_{K_\sigma}(x) = -\phi_{K_\sigma}(x) = \phi_{K_{-\sigma}}(x) \end{aligned}$$

This is typical for systems with spontaneous symmetry breaking, since the localized energy solutions connect the different vacua. That the (anti)kink is not invariant also under parity transformation (antisymmetric), will have interesting consequences for the fermionic boundary conditions in the quantum theory of supersymmetric solitons.

Energy density, classical kink mass By the use of the virial theorem (13) and the (anti)kink-solution one obtains for the energy density

$$\varepsilon(x)_{|static} = \frac{1}{2}\phi'^2 + U(\phi) = 2U(\phi) = \phi'^2 = \frac{\mu^4}{4\lambda}\frac{1}{\cosh^4[\frac{\mu}{\sqrt{2}}(x - x_0)]},$$

which is regular for real x and satisfies the conditions of the definition for solitary waves (see fig 3. So the kink and antikink are *solitary waves*. In analogy to the rest mass of a particle the total energy $E[\phi]$ for the (anti)kink is called the classical kink mass, since they are static. It is given by

$$M_{cl} = \int dx \varepsilon(x)_{|static} = \frac{\mu^4}{4\lambda}\int \frac{dx}{\cosh^4[\frac{\mu}{\sqrt{2}}(x - x_0)} = \frac{2\sqrt{2}\mu^3}{3\lambda}. \tag{22}$$

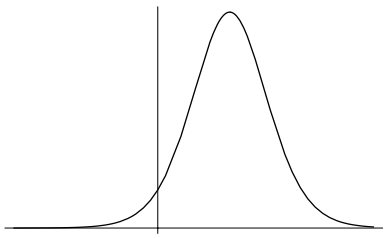


Figure 3: The energy density for the kink and antikink. It is localized around the center x_0 of the (anti)kink an characterized by its half-width $d_{\frac{1}{2}} = \frac{1}{\mu}$.

2.3.2 Moving kinks

By a 2D-Lorentz boost to a system moving with velocity u one obtains a moving (anti)kink. Since ϕ is a scalar field we only have to transform the coordinate:

$$x \to \gamma(x - ut) = \frac{x - ut}{\sqrt{1 - u^2}}.$$

In these new coordinates the kink reads as follows

$$\phi_u(x,t) = \frac{\mu}{\sqrt{\lambda}} \tanh[\frac{\mu}{\sqrt{2}} \frac{(x - x_0) - ut}{\sqrt{1 - u^2}}] \ , \ u \in (-1, 1) \tag{23}$$

It is easy to check that this is a solution of the equation of motion (17). The energy density and kink mass are obtained by inserting (23) into (6):

$$\varepsilon(x,t) = \frac{2}{1 - u^2}[\frac{\mu^4}{4\lambda} - \frac{\mu^2}{2}\phi_u^2 + \frac{\lambda}{4}\phi_u^4].$$

For the total energy this gives with a variable substitution $x \to \frac{\mu}{\sqrt{\lambda}} \frac{(x-x_0)-ut}{\sqrt{1-u^2}}$ in the integral of (22):

$$E[\phi_u] = \frac{2\sqrt{2}\mu^3}{4\lambda\sqrt{1 - u^2}} \int dx[\frac{1}{\cosh^2 x} - \frac{\sinh^2 x}{\cosh^4 x}] = \frac{1}{\sqrt{1 - u^2}} \frac{2\sqrt{2}\mu^3}{3\lambda} = \frac{M_{cl}}{\sqrt{1 - u^2}}.$$

This is a very nice result, since it is the relativistic energy-mass-relation for a particle. So one can expect that the quantized theory will give one particle states associated with the classical extended object.

2.4 Topological indices and topological conservation laws

It is often possible to make a topological classification of the solutions of a system. Specifically if one can define a topological index which is conserved in time it will play an important role of a quantum number in the quantum theory as other conserved quantities. But its origin is different from that of familiar conserved quantities. Again we consider a special class of theories in 2D, namely

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - U(\phi),$$

where $U(\phi)$ has a *discrete* set of absolute minima $U(\phi)_{min} = 0$. We are interested in nonsingular solutions with *finite energy* which also include solitary waves and solitons. The requirement of finite energy at any time t_0 implies the following boundary conditions:

$$\lim_{x \to \pm\infty} \phi(x, t_0) \equiv \phi(\pm\infty, t_0) = \phi_\pm \tag{24}$$

where at $\phi_\pm$ the potential U has an absolute minimum, i.e. $U(\phi_\pm) = 0$. Since this must be true for all t_0 (all terms in $E[\phi]$ are positive, see (6)) and the discrete minima of U are separated, the field $\phi(\infty, t) = \phi_\pm$ must be *stationary*, i.e. conserved:

$$\partial_t \phi(\pm\infty, t_0) = 0. \tag{25}$$

Thus we can divide the space of all non-singular finite-energy solutions into sectors, characterized by two *time independent indices,* namely $\phi(-\infty)$ and $\phi(\infty)$. These sectors are *not* topological connected. Fields of one sector cannot be deformed continuously into fields of another sector without violating the finite-energy-condition. This emphasizes the difference between the conserved indices (25) and familiar conservation laws which are a consequences of continuous symmetries of the theory. The fact that different sectors are not connected is a consequence of the topological property of the space of non-singular finite energy solutions. For this reason (25) is called a topological conservation law.

One can show that the existence of a topological conservation law is sufficient for the existence of non-dissipative solutions. This is important in more complicated theories, for which the direct integration is not so easy as for a single scalar field in $D = 1 + 1$ dimensions (this will be shortly discussed in section 2.7). By means of the ϕ^4 model we want give an idea how this works. Instead of nonsingular solutions of finite energy we consider non-singular initial-value data $\phi(x, t_0)$ and $\partial_t \phi(x, t_0)$ at some fixed time t_0 (for the existence of nonsingular solutions for this initial value problem we refer to the reference in [6]). For these initial-value data, just as for the time-independent solutions (the kinks) the finite energy condition implies the relations (24) and (25). If U has more than one absolute minimum equation (25) is non-trivial. Now one can show [6] that any solution with the conserved indices

$$\phi(\infty, t) = -\phi(-\infty, t),$$

is non-dissipative, i.e. that the energy density does not spread indefinitely with time ($\lim_{t\to\infty} \max_x \varepsilon(x, t) \neq 0$). By continuity in x, for any t, there must be some x_0 for which $\phi(x_0, t) = 0$. At this point the energy density (6) is

$$\varepsilon(x_0, t) \geq U(0) = \frac{\mu^4}{4\lambda}.$$

Thus for all times the maximum of the energy density is unequal zero,

$$\max_x \varepsilon(x, t) \geq \frac{\mu^4}{4\lambda},$$

and therefore the energy density does not dissipate but stays localized. In an analogous way the existence of nontrivial topological conservation laws can be used to prove the existence of non-dissipative solutions.

Topological indices of the (anti)kink

The potential $U(\phi) = 1/4\lambda(\phi^2 - \mu^2/\lambda)^2$ has two minima at $\phi_{min} = \pm\mu/\sqrt{\lambda}$. This gives rise to four topological sectors of non-singular finite-energy solutions with the following indices set (writing only the signs)

$$\{(\phi(\infty), \phi(-\infty))\} = \{(-,+),(+,-),(-,-),(+,+)\}.$$

The kink, the antikink and the two trivial solutions $\phi = \pm\frac{m}{\sqrt{\lambda}}$ are elements of the four sectors respectively. Another example is a kink from $x \to -\infty$ and a antikink from $x \to \infty$ approaching each other. This field configuration lies in the (trivial) $(-\frac{m}{\sqrt{\lambda}}, -\frac{m}{\sqrt{\lambda}})$ sector. Even though one cannot easily calculate the collision, we know that the resulting field will always stay in that sector. In fact the (anti)kink is only a solitary wave and not a soliton. These topological constraints also stabilize the (anti)kink and because of this these nontrivial solutions will become fundamental particles in the quantum theory, since they cannot decay. For a decay the (anti)kink would be deformed into a trivial topological sector, which would need an infinite amount of energy.

This fact, that the (anti) kink cannot be deformed continuously into the trivial sector without violating the finite energy condition is the origin of misunderstandings in the use of boundary conditions in the trivial and nontrivial sector during the quantization procedure. Also because of this one temporarily uses the kink-antikink configuration for the quantization procedure which is an intractable trick in more complicated cases ([31],[32],[46]). This will be clarified up later.

2.4.1 Topological charge and conserved current

Although the conserved topological indices come from the finite-energy-condition and not from a continuous symmetry, one can define a conserved current and a corresponding charge connected to the topological indices

$$Q := c\,[(\phi(x=\infty) - (\phi(x=-\infty)]\ ,\ k^\mu := c\,\varepsilon^{\mu\nu}\partial_\nu\phi \tag{26}$$

where $\varepsilon^{\mu\nu}$ is the antisymmetric epsilon symbol and c is an arbitrary constant. This is trivial in D=1+1. With these definitions one has

$$\partial_\mu k^\mu = 0 \text{ and } Q = \int dx k_0. \tag{27}$$

Note that plane waves $e^{ip_\mu x^\mu}$ do not change the topological charge. Assume a field configuration ϕ_{top} with a definite topological charge Q_{top}, like the kink or the vacuum. Then the topological current k^μ in (26) and the topological charge Q (27) get an additional contribution from the plane wave for the field $\phi_{top} + e^{ip_\mu x^\mu}$ as follows

$$\delta k^\mu = \varepsilon^{\mu\nu}\partial_\nu(e^{ip_\mu x^\mu}) = \varepsilon^{\mu\nu} i p_\nu (e^{ip_\mu x^\mu}) \tag{28}$$

$$\delta Q = ip_1\, e^{ip_0 t}\int dx\, e^{-ip_1 x} = p_1\delta(p_1)2\pi i\, e^{ip_0 t} = 0. \tag{29}$$

Thus small (quantum) fluctuations will not change the topological charge and the topological sector of the classical finite energy field configuration with a certain topological charge.

To classify the topological sector one needs $\phi(x=\infty)$ and $\phi(x=-\infty)$, so that the knowledge of Q is not enough, but for quantities which depend only on the difference of the conserved indices Q is sufficient.

For our ϕ^4-theory we set $c=\sqrt{\lambda}/m$ so that $Q \in \{-1,0,1\}$. Solitary waves are called *topological* if $Q \neq 0$, otherwise *non-topological*. This terminology says that the nontrivial solutions of the ϕ^4-theory are topological.

Symmetry breaking and topological indices. Suppose the Lagrangian $\mathcal{L}$ is invariant under some transformation T of the fields. Then one can distinguish two cases:

1. U has a unique minimum at $\phi_0 \Rightarrow \phi_0$ itself must be invariant under T, i.e. $T\phi_0 = \phi_0$, since a symmetry transformation does not change the energy of a solution.

2. U has several degenerated minima at ϕ_i , $i=1,...M>1 \Rightarrow$ the full set $\{\phi_i\}$ must be invariant under T, i.e. $T\phi_i \in \{\phi_i\}$, but not each ϕ_i itself. If not each ϕ_i itself is invariant under T one calls this a *spontaneously broken symmetry* (see above).

In order to get non-trivial topological sectors the existence of more than one degenerated minimum of U is necessary and sufficient (see ref. in [6] for the existence of nonsingular solutions for nonsingular initial-value data of finite energy). Thus a spontaneous symmetry breaking gives rise to nontrivial topological sectors. The converse is not always true, since the ϕ_i's could nevertheless be invariant under T.

2.5 The sine - Gordon system in D=1+1

The ϕ^4 - theory discussed above yields only solitary waves, but not solitons. The sine - Gordon system also yields solitons, as we will see. The sine-Gordon equation has a long story. In the last quarter of the nineteenth century it was extensively studied by geometers since it describes a two dimensional Riemann-surface of constant negative Gaussian curvature $K=-1$ [8]. It entered particle physics through works of Skyrme (1958,1960) who studied simple nonlinear field theories. Its name is a pun and seems to belonging to either Finkelstein and Rubinstein (Klein-Gordon $\rightarrow$ sine-Gordon) or Kruskal who investigated numerical solutions of nonlinear field equations and also discovered solitonic solutions of the Korteweg-de Vries equation (see [9] and references therein. The history of this name is not completely clear but it has prevailed over other names). A mechanical system which is also described by the sine-Gordon equation is realized by a continuous chain of elastic connected pendular on a horizontal line in a constant gravity field (or an infinite ribbon with a load at one edge [9]). The field $\phi(x,t)$ in this case describes the angular amplitude of the pendulum. The sine-Gordon equation also describes an infinite Jefferson contact [10]. As one can see a lot of systems are described by the sine-Gordon equation.

The Lagrangian of the sine-Gordon system is given by

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{\mu^2}{\gamma}\left[\cos(\sqrt{\gamma}\phi) - 1\right].$$

A series expansion shows that approximations of this Lagrangian are well known systems

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}\mu^2\phi^2 + \mu^2\frac{\gamma}{4!}\phi^4 + O(\gamma^2).$$

For $\gamma \to 0$ this is just the free Klein - Gordon field and including the $O(\gamma)$ - term one has the (attractive) ϕ^4 - theory. The equations of motions are

$$\Box\phi + \frac{\mu^2}{\sqrt{\gamma}} \sin(\sqrt{\gamma}\phi) = 0.$$

With a change in variables $x^\alpha \to \frac{1}{\mu}x^\alpha$ and the rescaling $\phi \to \frac{1}{\sqrt{\gamma}}\phi$ the Lagrangian and equation of motion writes as

$$\mathcal{L} = \frac{\mu^2}{\gamma}[\frac{1}{2}(\partial\phi)^2 + \cos\phi - 1] \tag{30}$$

$$\Box\phi + \sin\phi = 0. \tag{31}$$

As one can see, in principle the system can be described by only one parameter $\beta := \frac{\mu^2}{\gamma}$, which does not enter the classical e.o.m. It is a generic property that the classical field equations are independent of the coupling γ. Also for the ϕ^4 theory (15) this can be achieved by rescaling the field as $\bar{\phi} = \lambda\phi$. This can also be seen by the fact that in classical physics γ (λ) is a dimensionful parameter and thus can be used to set the scale. Of course, γ (λ) *is* relevant in quantum physics, since quantum physics contains a new constant, $\hbar$, and the important object in quantum theory is $\frac{1}{\hbar}\mathcal{L}$ and the relevant dimensionless parameter is $\frac{\hbar}{\beta}$ or $\hbar\lambda$, respectively (see section. 3).

Energy and energy density are given by (note that $dx \to 1/\mu\, dx$)

$$E[\phi] = \int dx\varepsilon(x,t) = \int dx\frac{\mu}{\gamma}[\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\phi'^2 + (1-\cos\phi)] \tag{32}$$

The Lagrangian $\mathcal{L}$ and so the equations of motion have the following discrete symmetries

$$\phi \to -\phi \text{ and } \phi \to \phi + 2\pi N, N \in \mathbb{Z} \tag{33}$$

Since the overall factor μ^2/γ does not enter the classical e.o.m. (31) we consider the potential

$$U(\phi) = (1 - \cos\phi),$$

whose minima are given by (note that $U \geq 0$)

$$\frac{dU}{d\phi} = \sin\phi \stackrel{!}{=} 0 \Rightarrow \phi_M = 2\pi M, M \in \mathbb{Z}$$

Thus we have a countably infinite set of absolute minima for which the energy $E[\phi_M]$ vanishes, since $U(\phi_M) = 0$. The minima ϕ_M are transformed into each other by the discrete symmetries (33) and therefore this symmetry, except for a $\mathbb{Z}_2$ transformation, is also spontaneously broken.

For finite energy - solutions, i.e. $\lim\limits_{x\to\pm\infty}\varepsilon(x,t) = 0$, we get the following boundary condition

$$\phi(x \to \pm\infty) = 2\pi M_\pm \tag{34}$$

Thus, according to (34) we can characterize our topological sectors by the conserved pair of integer indices (M_+, M_-). If only fields modulo 2π are physically relevant (this depends on the "interpretation" of the particle states [6]), then only the topological charge

$$Q \equiv M_+ - M_- = \frac{1}{2\pi}\int dx\partial_x\phi$$

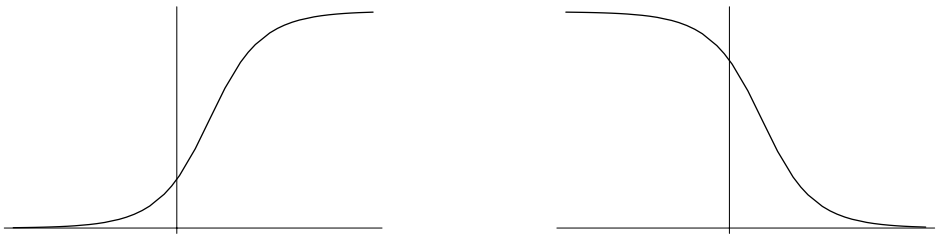


Figure 4: The "fundamental" ($k = 1$) soliton ϕ_{S_+} and antisoliton ϕ_{S_-} of the sine-Gordon system.

matters.

From our analysis of the mechanical analogue we know that a nontrivial finite-energy solution must move with x from one absolute minimum of U to a neighboring one, i.e. they must carry the charge $Q = \pm 1$. From (14) we get the static solution as follows:

$$x - x_0 = \pm \int_{\phi(x_0)}^{\phi(x)} \frac{d\phi}{\sqrt{2(1-\cos\phi)}}$$

with $\frac{1}{2}(1-\cos y) = \sin^2\frac{y}{2}$ and setting the integration constant $\phi(x_0) = \pi$ one obtains by solving for[3] ϕ

$$\phi_{S_\pm}(x) = 4\arctan[e^{\pm(x-x_0)}] + 4k\pi \ , \ k \in \mathbb{Z} \tag{35}$$

The solution ϕ_{S_+} with the $(+)$ sign is called the *soliton*, the solution ϕ_{S_-} with $(-)$ sign is called the *antisoliton* of the system. Their graphs are plotted in fig.4 and are very similar to the kink and antikink of the ϕ^4 model (for both models we will often call them simply kinks ϕ_K). As one can see, for both, ϕ_S and ϕ_{S_-}, there exists an infinite set of solutions which connect different neighboring minima ϕ_M of the potential. Their topological charges are

$$Q_{S_+} = \frac{1}{2\pi}[\phi_{S_+}(x=\infty) - \phi_{S_+}(x=-\infty)] = (2k+1) - 2k = 1$$
$$Q_{S_-} = \frac{1}{2\pi}[\phi_{S_-}(x=\infty) - \phi_{S_-}(x=-\infty)] = 2k - (2k+1) = -1.$$

For the mechanical realization this solution describes a chain of pendular which is turned around once at the position x_0. The sign of the topological charge describes the orientation of the winding and is opposite for soliton and antisoliton. one obtains the classical mass of each (anti)soliton using the virial theorem (13) and (32) :

$$M_{cl}^S = \int dx\varepsilon(x)_{|static} = \frac{\mu}{\gamma}\int dx[\frac{1}{2}\phi'^2 + U] = \frac{4\mu}{\gamma}\int dx\frac{1}{\cosh^2(x-x_0)} = \frac{8\mu}{\gamma} \tag{36}$$

2.5.1 Time dependent solitons

Again one obtains moving solutions by a 2D Lorentz-boost, i.e. a coordinate transformation (the γ used here has nothing to do with coupling, it is the usual parameter of relativistic kinematics):

$$x \to \gamma(x-ut) = \frac{x-ut}{\sqrt{1-u^2}}.$$

[3] In the "old" literature one uses different branches of the inverse tangens. We follow [7] and use the unique arctan to avoid misunderstandings.

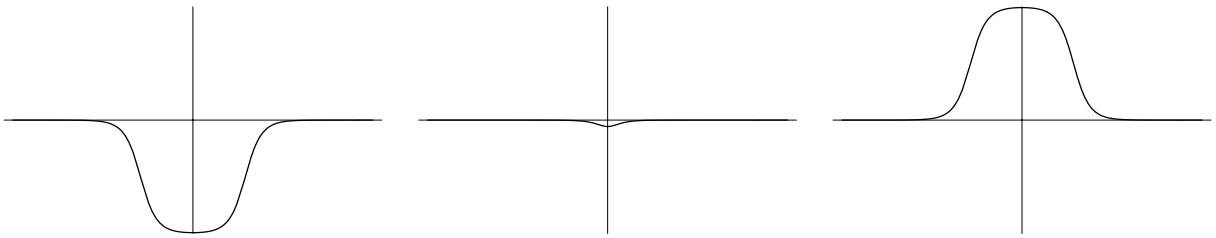


Figure 5: Soliton-antisoliton scattering at the times $t << 0$, $t \approx 0_-$ and $t >> 0$

From this we get the moving (anti)soliton

$$\phi_{S_\pm,u}(x,t) = 4\arctan[e^{\pm\gamma(x-x_0-ut)}] + 4k\pi \ , \ k \in \mathbb{Z}.$$

We already called these solutions (anti)solitons because the sine - Gordon system provides several soliton-solutions according to our definition of solitons. We only can mention some examples here:

Soliton-antisoliton-scattering. This solution of (31) is

$$\phi_{SA}(x,t) = 4\arctan[\frac{u\cosh(\gamma x)}{\sinh(\gamma ut)}] = 4\arctan[\frac{\sinh(\gamma ut)}{u\cosh(\gamma x)}] \pm 2\pi$$

where γ is defined above and the sign in the second form depends on the quadrant of the argument of arctan. That this is an exact solution can be proved by insertion in the e.o.m. The asymptotic behavior shows that this is a soliton solution (with the abbr. $\delta = \frac{\ln u}{\gamma u}$):

$$\begin{aligned}
\phi_{SA}(x,t) \quad &\underset{t\to-\infty}{\longrightarrow} 4\arctan[2u\cosh(\gamma x)e^{\gamma ut}] \\
&\underset{x\to\infty}{\longrightarrow} 4\arctan[e^{\gamma(x+u(t+\delta))}] \qquad = \phi_{S_+}(\gamma[x+u(t+\delta)]) \\
&\underset{x\to-\infty}{\longrightarrow} 4\arctan[e^{-\gamma(x-u(t+\delta))}] \qquad = \phi_{S_-}(\gamma[x-u(t+\delta)])
\end{aligned}$$

Thus for $t \to -\infty$ we have a soliton moving with velocity $-u$ from $x \to \infty$, i.e. to the center $x = 0$, and a antisoliton moving with velocity u from $x \to -\infty$, i.e. an soliton and antisoliton approaching each other. For the asymptotic future we have

$$\begin{aligned}
\phi_{SA}(x,t) \quad &\underset{t\to\infty}{\longrightarrow} 4\arctan[\tfrac{1}{2u}\tfrac{e^{\gamma ut}}{\cosh(\gamma x)}] + 2\pi \\
&\underset{x\to\infty}{\longrightarrow} 4\arctan[e^{-\gamma(x-u(t-\delta))}] + 2\pi \quad = \phi_{S_-}(\gamma[x-u(t-\delta)]) \\
&\underset{x\to-\infty}{\longrightarrow} 4\arctan[e^{\gamma(x+u(t-\delta))}] + 2\pi \quad = \phi_{S_+}(\gamma[x+u(t-\delta)])
\end{aligned}$$

which are again a soliton and antisoliton with the same shape and velocities, but now departing from each other. The only change from the initial condition is the time delay δ which remains the sole residual effect of the collision. Since in our units $u < 1$, the delay δ is negative. This indicates that the soliton and antisoliton attract each other. This can be seen very illustrative by the mechanical realization, where the two different windings attract each other.

At $t = 0$, ϕ_{SA} vanishes (the two opposite windings come together and unwind each other), i.e. the approaching (anti)solitons tend to annihilate each other until $t = 0$, but the field re-emerges for growing t and asymptotically restore the soliton and antisoliton (fig.5). Since

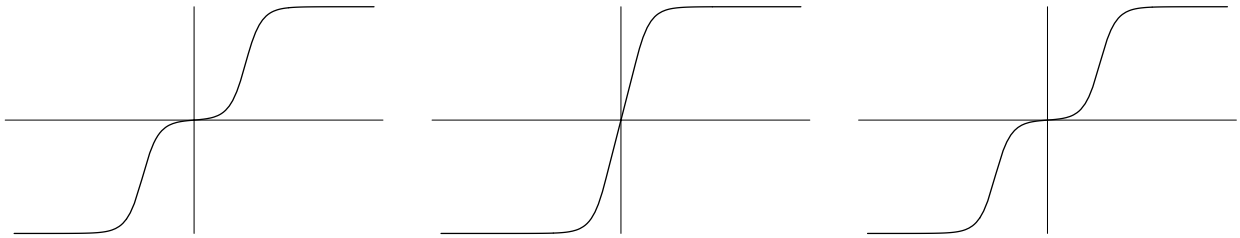


Figure 6: The soliton-soliton scattering at $t < 0$, $t = 0$, i.e. their maximum approach and $t > 0$. After this they disappear from each other again in the opposite direction of their approach.

every solution of the sine-Gordon system is a solution modulo 2π the graph can be shifted up and down by 2π-steps.

Soliton-soliton scattering. This solution is given by

$$\phi_{SS}(x,t) = 4\arctan[\frac{u\sinh(\gamma x)}{\cosh(\gamma u t)}]$$

An analogous procedure as above shows that asymptotically two solitons approach each other for $t \to -\infty$ and departing from each other with same shape and speed, but with opposite velocity and a time delay. Thus they bounce back (backward scattering). At any instant of time the field ranges from -2π to $+2\pi$ as x goes from $-\infty$ to ∞. So the solution lies in the $Q = 2$ -sector (= total winding number of the pendular-chain). If we do not distinguish between fields modulo 2π, then there is no difference asymptotically between backward and forward scattering (see fig.6). The discrete symmetry under $\phi \to -\phi$ gives us an analogous solution for two antisolitons, namely $\phi_{AA} = -\phi_{SS}$.

The doublet or breather solution. This solution is obtained by setting $u = \mathrm{i}v$ in the solution ϕ_{SA} (omitting the modulo-constant 2π)

$$\phi_v(x,t) = 4\arctan[\frac{\sin(\frac{vt}{\sqrt{1+v^2}})}{v\cosh(\frac{x}{\sqrt{1+v^2}})}] \tag{37}$$

this is still a real exact solution of sine-Gordon system. The parameter v is now not a velocity but it is connected to the period τ of this periodic solution

$$\tau = 2\pi\frac{\sqrt{1+v^2}}{v}.$$

The breather solution can be considered as a soliton and antisoliton oscillating about each other (see fig.7). In contrast to the scattering solution ϕ_{SA} the soliton and antisoliton input does not separate arbitrarily far apart as $t \to \pm\infty$ but rather separate only up to a finite distance and never become fully free and undistorted. Thus this can be seen as a bound solution. The solution is given in its rest frame, i.e. it is centered around $x = 0$ for all time. Moving breather-solutions are again obtained by a 2D-Lorentz boost. But also in its rest frame the breather has nontrivial time dependence in contrast to the former solutions which are static in their rest frame. The breather does not fit into our definition of solitary waves but the field and the energy density is localized.

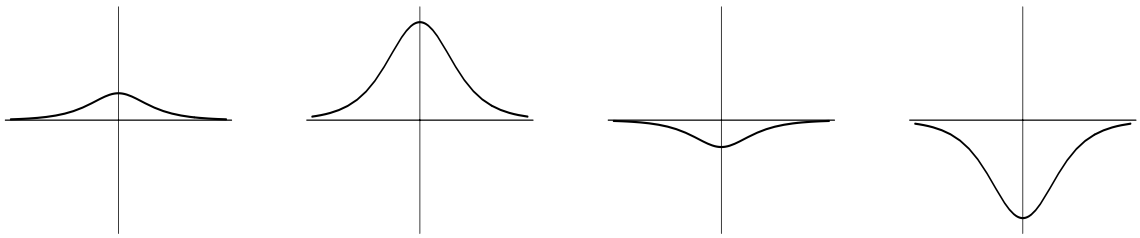


Figure 7: The famous sine-Gordon breather for different times $t_1 > t_2 > t_3 > t_4$. It is periodic in time with the period τ.

2.6 Stability and zero modes

2.6.1 Static non-trivial solutions

We have already mentioned that the static finite energy solutions will become new fundamental particles(states) in the quantum theory. Therefore we investigate the stability of these objects to see if they "survive" the quantization procedure. We have already seen that the existence of a topological conservation law leads to non-dissipative solutions, i.e. solutions for which the energy density does not dissipate to zero. Now we investigate the solutions explicitly according to their behavior under small perturbations. The equations of motion for our models are

$$\Box\phi + U'(\phi) = 0. \tag{38}$$

Next we consider a small perturbation of a static solution $\phi_{cl}(x)$ of this e.o.m.

$$\phi(x,t) = \phi_{cl}(x) + \eta(x,t), \tag{39}$$

and investigate the development with time of the perturbation $\eta(x,t)$, which is determined by e.o.m. Inserting (39) into (38) one gets the following e.o.m for the perturbation η:

$$\Box\eta + U''[\phi_{cl}(x)]\eta = 0 + O(\eta^2). \tag{40}$$

The linearized e.o.m. for the perturbation is separable for static solutions (invariant under time translations) and the general solution for η is a superposition of normal modes:

$$\eta(x,t) = \mathrm{Re}\sum a_n\, e^{i\omega_n t}\xi_n(x) \tag{41}$$

The sum has to be treated as an integral for continuum modes. The coefficients a_n are arbitrary complex numbers and the frequencies ω_n and modes ξ_n have to solve the eigenvalue problem (inserting the ansatz (41) into the linearized equation (40)):

$$\left(-\partial_x^2 + U''(\phi_{cl})\right)\xi_n = \omega_n^2\xi_n. \tag{42}$$

This is a Schrödinger equation with the potential $U''(\phi_{cl})$ and in this context called *stability equation*. If one can explicitly solve this equation, as it is possible for the SG - and ϕ^4- model, one obtains (in the linear approximation) a rich set of solutions of the e.o.m. In (41) one can make the following classification for the perturbation w.r.t. the eigen-values ω_n^2 of (42):

$\omega_n^2 > 0$: The mode ξ_n stays oscillatory in time.

$\omega_n^2 < 0$: The mode ξ_n grows exponentially with time and thus also the perturbation, the linearized equation is only valid for short times.

$\omega_n^2 = 0$: The mode ξ_n is a so called *zero mode,* the perturbation is constant in time.

First of all we can see that the static solution $\phi_{cl}(x)$ can only be stable if all eigenvalues of the stability equation (42) are positive, i.e. $\omega_n \geq 0$ for all n. The occurrence of zero modes is connected with symmetries of the system. In our special case it is the translation symmetry of the ϕ^4 and the SG- kinks[4]. The explicit solution of the stability equation of the ϕ^4 and the SG- kink are given in the appendix (section 7.1). In general one can show that because of the spatial translation invariance all eigenvalues are positive and therefore the solution ϕ_{cl} is stable [6]. Because of the spatial translation invariance, also the boundary conditions (7,8) are translational invariant; also $\phi_{cl}(x + x_0)$ is a solution if $\phi_{cl}(x)$ is one. For infinitesimal translations one has

$$\phi_{cl}(x + x_0) = \phi_{cl}(x) + x_0 \partial_x \phi_{cl}(x) + O(x_0^2),$$

inserting this into the e.o.m. (38) one gets as zero mode of the stability equation (42), i.e. the eigen-function to the eigen-value $\omega^2 = 0$, the function $\partial_x \phi_{cl}(x)$. As mentioned above the static finite energy solutions ϕ_{cl}, connecting neighboring minima of the potential $U(\phi)$, are monotonic functions. Therefore the derivative $\partial_x \phi_{cl}$ has no nodes. It is well known that for one-dimensional Schrödinger equations with arbitrary potentials the eigen-function with no nodes is the eigen-function with the lowest energy. Since this eigenvalue is equal to zero, all eigenvalues ω^2 are positive, and thus the perturbation η stays oscillatory in time and the static finite energy solutions are stable.

The occurrence of the zero mode will cause troubles in the quantization procedure. Another way of writing the stability-operator in (42) and (40) is to express it through the action $S[\phi] = \int dtdx\mathcal{L}$. For the classical solution $\phi_{cl}(x)$ the action is stationary, thus the e.o.m. writes as:

$$\frac{\delta S}{\delta \phi}|_{\phi_{cl}} = 0. \tag{43}$$

Expanding the action around ϕ_{cl} one obtains[5]

$$S[\phi_{cl} + \eta] = S(\phi_{cl}) + \frac{1}{2} \int dtdx\ \eta \left[\frac{\delta^2 S}{\delta \phi^2}|_{\phi_{cl}} \right] \eta + O(\eta^3). \tag{44}$$

The linear term is absent because of the e.o.m. (43) and the spatial part of the operator $\frac{\delta^2 S}{\delta \phi^2}|_{\phi_{cl}}$is exactly the operator in (42). The second term in (44) will be the central object in the quantization procedure.

To see that the occurrence of zero modes is in general connected with symmetries we consider a very general theory with an arbitrary number of fields $\{\phi_i\}$:

$$\mathcal{L} = \mathcal{A}_{ij}^{\mu\nu} \partial_\mu \phi_i \partial_\nu \phi_j + \mathcal{B}_{ij}^{\mu} \phi_i \partial_\mu \phi_j - U(\phi_i). \tag{45}$$

[4]Here we use as mentioned above for both models the notion kink for the static finite energy solutions.

[5]For the moment we neglect subtleties connected with surface terms. For the exact treatment see section 3.

For special constant matrices $\mathcal{A}_{ij}^{\mu\nu}, \mathcal{B}_{ij}^{\mu}$ and potentials $U(\phi_i)$ this theory also includes fermions and gauge fields . The equations of motion are given as

$$\mathcal{D}_{ij}\phi_j + \frac{\partial U(\phi_k)}{\partial \phi_i}\phi_i = 0, \tag{46}$$

where $\mathcal{D}_{ij}$ is the matrix-valued differential operator, including first and/or second order derivatives in general, of the coupled system (46). For fermions (Grassmann fields) one has to take care of signs of course. Assume that the field "vector" $\{\phi_i^{cl}(x)\} = \vec{\phi}^{cl}$ is a static solution of the e.o.m. (46) for certain boundary conditions, i.e.

$$\mathcal{D}_{ij}\phi_j^{cl} + \frac{\partial U(\phi_k)}{\partial \phi_i}|_{\phi^{cl}}\phi_i^{cl} = 0 \text{ with } \vec{\phi}^{cl}(\pm\infty) = \vec{C}_{\pm},$$

where $\vec{C}_{\pm}$ is some constant vector in the field-space. Now assume the existence of a continuous (internal or space-time) symmetry $\mathcal{R}$ of the theory (45) which does not involve the time and leaves the boundary conditions invariant. Since under a symmetry transformation the e.o.m. are invariant, the field configuration $\tilde{\phi}_i = \mathcal{R}\phi_k^{cl}$ is also a static solution. Since $\mathcal{R}$ is a continuous symmetry it is also possible to consider infinitesimal transformations. By an infinitesimal transformation one obtains a static solution of the form

$$\tilde{\phi}_i = \mathcal{R}\phi_k^{cl} = \phi_i^{cl} + \delta_{\mathcal{R}}\phi_k^{cl}.$$

Since $\tilde{\phi}_i$ is a classical solution, the deviation $\delta_{\mathcal{R}}\phi_k^{cl}$ fulfills the full stability equation (40), and since $\tilde{\phi}_i$ and ϕ_i^{cl} is static also the deviation $\delta_{\mathcal{R}}\phi_k^{cl}$ must be static and therefore the frequency of this mode in (41) must be zero. Thus $\delta_{\mathcal{R}}\phi_k^{cl}$ is a *zero mode* of the Schrödinger equation (42). The invariance of the boundary condition $\vec{C}_{\pm}$ ensures that $\tilde{\phi}_i$ is also a finite energy solution. This restriction is not really needed since because of the topological conservation law it is impossible for a continuous (symmetry) transformation to change the topological sector of a field configuration. If there are more, independent symmetries present which fulfill the above requirements, then each of them has its own zero mode. We call two symmetries independent if the Poisson bracket of their Noether charges vanishes. It is clear that in the presence of such a symmetry one does not only have one classical solution but a continuous set of such classical solutions.

If the classical solution ϕ_i^{cl} is an absolute minimum of the potential $U(\phi_i)$, i.e. the trivial (vacuum) solution, the associated "zero-mode" $\delta_{\mathcal{R}}\phi_k^{cl}$ is the *Goldstone mode*, which is connected with spontaneous symmetry breaking and arises from the "valley" of continuous degenerated vacua. We want to distinguish this case from the occurrence of zero eigen-values connected with nontrivial solutions. Therefore we reserve the notion zero modes for the nontrivial case.

For our static kinks the continuous symmetry is the space-translation invariance and the continuous family of solutions is parametrized by the position (center) of the kink x_0. As can be seen in the appendix this leads to a zero mode and the fact that the classical solution is not isolated will spoil some requirements in the quantization procedure (see below).

2.6.2 Periodic time dependent solutions

For time-dependent solitons things are not as simple as for the static ones and therefore one needs somewhat more advanced mathematical techniques. We consider non-trivial solutions

$\phi_T(x,t)$ which are periodic in time, even in their rest-frame, with the period T:

$$\Box\phi_T + U'(\phi_T) = 0 \text{ with } \phi_T(x,t+T) = \phi_T(x,t). \tag{47}$$

For a small perturbation $\eta(x,t)$ the linearized e.o.m. again gives the stability equation

$$[\Box + U''(\phi_T)]\,\eta(x,t) = 0, \tag{48}$$

but now the stability operator in (48) is no longer separable (invariant under time translations), since the potential $U''(\phi_T)$ is now time-dependent through $\phi_T(x,t)$. Because of the periodicity of ϕ_T the stability equation (48) is periodic with the same period T and thus invariant under time translations $t \to t+T$. Therefore (48) is the field theoretical analogue of the *Hill* equation, known from point-mechanics in connection with the stability of periodic orbits [11]. Because of the residual time translation invariance the solutions of the stability equations have special properties, described by *Floquet's* theorem [13]. The solutions of the stability equation are of the form (no summation over indices)

$$e^{\pm i\mu_n t}\xi_n(x,t) \text{ with } \xi_n(x,t+T) = \xi_n(x,t) \tag{49}$$

where T is the period of the stability operator. Because of the reality of the stability equation the solutions come in complex conjugated pairs. A perturbative mode (49) after a time T has the form

$$\eta_n(x,t+T) = e^{\pm i(\mu_n t + \mu_n T)}\xi_n(x,t) =: e^{\pm i\nu_n}\eta_n(x,t)$$

The phases $\nu_n := \mu_n T$ are called *stability angles*[24], *characteristic exponents* [13] or *phase advance* [11], and in general consist of a discrete and a continuous set, like the eigenvalues ω_n^2 in the static case. They are the generalized analogue of the frequencies ω and characterize the periodic "orbit" $\phi_T(x,t)$. If one knows all stability angles ν_n, this means to know all solutions of the stability equation, and one can decide for the stability of the periodic solution as follows:

ν_n = real:	The mode ξ_n stays oscillatory in time.
ν_n = complex:	The mode ξ_n grows exponentially with time and thus also the perturbation, the linearized equation is only valid for short times.
$\nu_n = 0$:	The mode ξ_n is a so called *zero mode,* the perturbation is constant in time.

This classification is quite analogous to the above one for static solutions. As one can see, for the classical solution $\phi_T(x,t)$ to be stable all stability angles ν_n must be real. Modes with vanishing stability angles $\nu_n = 0$ are called *zero modes* and their occurrence again leads to problems in the quantization procedure. As in the static case, these zero modes are connected with the symmetries of the system:

Assume the existence of a continuous symmetry $\mathcal{R}$ of the theory which leaves the e.o.m. *and* the boundary conditions (47) invariant. Thus if $\phi_T(x,t)$ is a solution of (47) then $\tilde{\phi}_T = \mathcal{R}\phi_T$ is also a solution, i.e.

$$\Box\tilde{\phi}_T + U'(\tilde{\phi}_T) = 0 \text{ with } \tilde{\phi}_T(x,t+T) = \tilde{\phi}_T(x,t).$$

Since $\mathcal{R}$ is a continuous symmetry there exists an infinitesimal transformation of ϕ_T which is again a solution of (47) and has the form

$$\tilde{\phi}_T(x,t) = \mathcal{R}\phi_T = \phi_T(x,T) + \delta_{\mathcal{R}}\phi_T.$$

Since $\tilde{\phi}_T$ is a classical solution the deviation $\delta_{\mathcal{R}}\phi_T$ fulfills the stability equation (48) and since $\tilde{\phi}_T$ and ϕ_T are periodic with the period T the deviation $\delta_{\mathcal{R}}\phi_T$ is also periodic with the period T. By *Floquets* theorem $\delta_{\mathcal{R}}\phi_T$ must be of the form (49) and since it is periodic with period T the phase μ must be zero. Thus $\delta_{\mathcal{R}}\phi_T$ has a zero stability angle $\nu = \mu T$ and is therefore a *zero mode*. For all independent symmetries fulfilling the above requirements there exists a separate zero mode. In the presence of such a symmetry there exists not an isolated but a continuous set of periodic solutions of periodicity T.

An illustrative example is the Kepler problem (the above statements are of course also true for discrete systems). Because of the rotational symmetry one can rotate the Kepler ellipse in the plane and each ellipse is a periodic solution with the same period and same angular momentum. So one has a continuous set of ellipses and not an isolated periodic orbit of given period. With the time translationally invariant classical e.o.m. and BC (47), the time derivative $\partial_t\phi_T(x.t)$ is always a zero mode.

As a summary one can say that if one can explicitly solve the stability equations one has a rich set of solutions of the (linearized) equations of motions. For the static case the explicit solutions for the ϕ^4- and SG- model are given in the appendix. For the periodic time-dependent SG- breather solution (37) this was done by Dashen, Hasslacher and Neveu in [24]. Of special interest are solutions with discrete eigenvalues ω^2 (or stability angles ν), other than the zero mode. The associated solution $\phi_{cl} + \eta$ yields a solution to the (linearized) e.o.m. which is of finite energy and periodic in time, also for static solitons. In quantum theory one can think of this situation as a meson bound to a soliton. From the continuum eigen-functions one can only form a solution of finite energy by forming a wave packet (this is possible for the linearized e.o.m.). In quantum theory this wave packet can be seen as a meson scattering off a soliton.

2.7 Existence of non-singular finite-energy solutions

An obvious extension of the models investigated above is to consider scalar field theories in more then one spatial dimension and perhaps with more than one scalar field. Unfortunately this does not lead to new static non-trivial solutions which is expressed in the no-go theorem [6]:

Derrick's theorem. Let $\vec{\phi} = \{\phi_i\}$ be a set of scalar fields in $D = 1 + d$ dimensions whose dynamics is described by

$$\mathcal{L} = \frac{1}{2}\partial_\mu\vec{\phi}\cdot\partial^\mu\vec{\phi} - U(\vec{\phi}),$$

and let U be positive and zero for the ground state(s) (minima) of the theory. Then the only non-singular time-independent solutions of finite energy are the ground states.

Proof. Define

$$V_1 = \frac{1}{2}\int d^d\mathrm{x}(\vec{\nabla}\phi)^2$$
$$V_2 = \int d^d\mathrm{x}U(\vec{\phi}).$$

Both functionals V_1 and V_2 are non-negative and are simultaneously zero only for the ground states. Let $\vec{\phi}(\mathrm{x})$ be a static solution. Consider the one-parameter family of field configurations

defined by

$$\vec{\phi}_\lambda(\mathrm{x}) := \vec{\phi}(\lambda \mathrm{x}),$$

where λ is a positive number. For this family the energy is given as

$$E_\lambda = \lambda^{(2-d)} V_1(\vec{\phi}_\lambda) + \lambda^{-d} V_2(\vec{\phi}_\lambda).$$

Since $\phi(\mathrm{x})$ is per assumption a solution oft the e.o.m., it follows by Hamilton's principle that the energy E_λ must be stationary at $\lambda = 1$. Thus,

$$(d-2)V_1(\vec{\phi}(\mathrm{x})) + dV_2(\vec{\phi}(\mathrm{x})) = 0.$$

For $d > 2$ this implies that both V_1 and V_2 vanish, which is only possible for the trivial ground states $\vec{\phi}(\mathrm{x}) = const.$ For two spatial dimensions only V_2 must vanish. Since U is per definition a positive function this implies again that $\vec{\phi}(\mathrm{x})$ is the trivial ground state for which U and thus V_2 vanishes, q.e.d.

Well, Derrick's theorem only denies the existence of static finite energy solutions in more than one spatial dimension. But these kind of solutions are of special interest since they appear as new particle states in the quantum theory, as mentioned above. For scalar fields the $D = 1 + 1$ dimensions are very special[6]. The existence of nonsingular finite energy solutions is also connected with the existence of topological conserved quantities (see above). In one spatial dimension there is "no way out" for the spatial asymptotic field values and thus leads to conserved topological indices.

A way to circumvent Derrick's no-go theorem is to consider gauge fields. For completeness we just give some comments[7], but for the rest of this work this topic is beyond our scope. As discussed in sect. 2.4, the existence of non-trivial finite energy solutions can be proved by the use of topological conservation laws and is connected with spontaneous symmetry braking. The main results for gauge theories in three dimensions are:

1. If the theory has no spontaneously broken gauge symmetry, the space of non-singular finite-energy solutions has only one component, and there are no non-trivial topological conservation laws.

2. The same situation as in (1) is present if the symmetry breakdown is total, i.e. if no massless gauge mesons survive.

3. If only one massless gauge boson survives (photon), the space of non-singular finite-energy solutions has an infinite number of components and there are non-trivial topological conservation laws, except when the gauge group contains a $U(1)$ factor whose generator enters into the expression of the electrical charge (e.g. Weinberg-Salam model).

4. Similar results as in (3) hold if many massless gauge bosons survive symmetry breakdown.

[6] 1+1 dimensions are of course in general very special.
[7] For more see for example [3] and [6].

Note that topological conservation laws enable us to establish the existence of non-dissipative solutions, not necessarily time-independent ones. Also topological conservation laws are sufficient but not necessary conditions for the existence of such solutions. It is quite possible that there exist non-dissipative solutions even when there are no non-trivial topological conservation laws. Nevertheless topological conservation laws give us important informations without exactly solving the e.o.m., which in higher dimensions is generally not as simple as for a static $D = 1 + 1$ scalar field, where it can be obtained by quadrature.

3 Quantization of Solitons

In standard quantum field theory the perturbative approach starts with solutions of the free field equation, i.e. solutions of linear equations. Quantum effects around these free solutions are calculated order by order. In the case of solitons we start even classically with solutions of the non linear equations and then quantize around these non-perturbative solutions. The appropriate formalism to implement any classical fields into the quantization procedure is the path integral. In standard perturbation theory, by this point of view, one quantizes around the trivial classical solution, i.e. the solution of lowest energy called the vacuum.

3.1 The path integral

To calculate the quantum corrections to the energy spectrum of classical nontrivial solutions we will use for static solitons the trace of the time-evolution operator or the "propagator". For periodic time-dependent solitons one uses the trace of the Green function (WKB method). Therefore we shortly review some fundamental relations.

3.1.1 Green functions, propagators and the spectral function

The time evolution of a quantum system is determined by the time dependent Schrödinger equation (Schrödinger picture)

$$\mathcal{H}\,|\psi(t)\rangle = i\hbar\partial_t\,|\psi(t)\rangle\,. \tag{50}$$

Equivalent to the knowledge of the states $|\psi(t)\rangle$ is the knowledge of the unitary *time-evolution operator* $U(t,t_0)$ of the system, which fulfills the initial data problem

$$\begin{aligned}(\mathcal{H} - i\hbar\partial_t)\mathcal{U}(t,t_0) &= 0\\ \mathcal{U}(t_0,t_0) &= \mathbf{1},\end{aligned}$$

and the *composition law*

$$\mathcal{U}(t'',t') = \mathcal{U}(t'',t)\mathcal{U}(t,t'). \tag{51}$$

Knowing $U(t,t_0)$ means having a solution of the time-dependent Schrödinger equation (50) in the sense that for a given initial state $|\psi(t_0)\rangle$, the state of the system at the time t is given by

$$|\psi(t)\rangle = \mathcal{U}(t,t_0)|\psi(t_0)\,|\psi(t_0)\rangle\,. \tag{52}$$

Closely related to the time-evolution operator is the *propagation kernel* (short: kernel) $\mathcal{K} = \theta(t-t_0)\mathcal{U}$, which fulfills the inhomogeneous equation

$$(\mathcal{H} - i\hbar\partial_t)\mathcal{K}(t,t_0) = -i\hbar\delta(t-t_0)\mathbf{1}$$
$$\lim_{t\to t_0}\mathcal{K}(t,t_0) = \mathbf{1} \quad .$$

Thus the kernel $\mathcal{K}$ is a Green-operator of the Schrödinger equation (50). Since $\mathcal{K}$ and $\mathcal{U}$ differ only by a step function, the following relations are very similar for $\mathcal{U}$. For time independent Hamiltonians[8] $\mathcal{H}$ for the kernel one immediately obtains the explicit solution

$$\mathcal{K}(t,t_0) = \theta(t-t_0)e^{-\frac{i}{\hbar}\mathcal{H}(t-t_0)}. \tag{53}$$

For time-dependent Hamiltonians $\mathcal{K}$ is a time-ordered product of infinitesimal versions of (53). For time-independent Hamiltonians, $\mathcal{K}$ or $\mathcal{U}$ only depends on the difference $T := t - t_0$. There are many systems for which it is more comfortable to work with the Fourier transformed kernel ($\epsilon > 0$)

$$\mathcal{G}(E) := \frac{i}{\hbar}\int dT\; e^{\frac{i}{\hbar}(E+i\epsilon)T}\mathcal{K} = \frac{i}{\hbar}\int_0^\infty dT\; e^{\frac{i}{\hbar}(E+i\epsilon-\mathcal{H})T} = \frac{1}{(\mathcal{H}-E-i\epsilon)}, \tag{54}$$

which fulfills the inhomogeneous Schrödinger equation

$$(\mathcal{H}-E)\mathcal{G}(E) = \mathbf{1}.$$

In mathematics, the operator $(\mathcal{A}-z)^{-1}$, $z \in \mathbb{C}\setminus \mathrm{spec}(\mathcal{A})$, is called the resolvent of a given operator $\mathcal{A}$. Therefore $\mathcal{G}$ is the resolvent kernel of $\mathcal{H}$ and its analytical structure gives the spectrum of $\mathcal{H}$. We have added a small imaginary part to the energy ("pole-prescription") to ensure convergence of the integral. The singularities of $\mathcal{G}$ are at $\epsilon = 0$, which means that the spectrum $\mathrm{spec}(\mathcal{H})$ of the Hamiltonian is real. In view of (54) $\mathcal{G}$ can equivalently be obtained by a Laplace transformation of the time-evolution operator $\mathcal{U}$.

Coordinate-representation. For notational simplicity and since the generalization to more degrees of freedom (DOF) is obvious, we first consider a one-dimensional quantum mechanical system, described by the Hamiltonian

$$\mathcal{H} = \frac{1}{2m}\hat{p}^2 + V(\hat{q}). \tag{55}$$

For practical calculations one works in the so called coordinate- or q- representation of the abstract Hilbert space. In this representation the spectrum of the operator $\hat{q}$ is defined as

$$\hat{q}\,|q\rangle = q\,|q\rangle \quad q\in\mathbb{R}, \text{ and } \langle q|\,\mathbf{1}\,|q'\rangle = \delta(q-q') \tag{56}$$

where we have assumed that the one-dimensional motion of the particle takes place on the whole real line, without additional topological constraints[9]. Because of the normalization in (56), both $\mathcal{K}$ and $\mathcal{U}$ fulfill first-order equations with distributional initial conditions. This

[8] In the Schrödinger picture $\mathcal{H}$ is always time-independent for fundamental theories. This not true for the Dirac picture.

[9] In the regularization procedure of the field theory we will consider such topological constraints.

representation of the kernel gives the (retarded) *Feynman* kernel also called *propagator* (this is not the "Feynman propagator" which occurs in field theory) as follows ($q' = q(t')$, $q'' = q(t'')$):

$$K(q'', T|q') = \theta(T) \langle q''| \mathcal{U}(t'', t') |q'\rangle = \theta(T) \langle q''| e^{-\frac{i}{\hbar}\mathcal{H}T} |q'\rangle \tag{57}$$

$$= \theta(T)_H \langle q'', t''|q', t'\rangle_H . \tag{58}$$

In the last equation we have expressed the kernel in terms of Heisenberg coordinate-states, which are eigen-states of the time-dependent Heisenberg operators

$$\hat{q}_H(t) |q, t\rangle_H = q_H(t) |q, t\rangle_H .$$

Thus $K(q'', T|q')$ is the amplitude that a particle starting at the position q' at the time t' is at the position q'' at the time t''. With (52), the time evolution of the Schrödinger wave function, defined by

$$\psi(q, t) := \langle q|\psi(t)\rangle \tag{59}$$

is given by the kernel as follows ($T = t'' - t' > 0$)

$$\psi(q'', t'') = \int dq' K(q'', T|q')\psi(q', t').$$

The composition law (51) implies for the kernel K ($t'' > t_1 > t'$)

$$K(q'', t''|q', t') = \int dq_1 K(q'', t''|q_1, t_1)K(q_1, t_1|q', t'),$$

the *law for the composition of amplitudes* for events which occur successively in time [22].

One can derive the path integral representation of the kernel (amplitude) K by multiple insertion of unity

$$\int dq(t_i) |q, t_i\rangle \langle q, t_i| =: \int dq_i |q_i\rangle \langle q_i|$$

between the Heisenberg states in (58) on a time-lattice $\{t_i\}$. With the abbreviation $t_{i+1} - t_i = \varepsilon$ and $N\varepsilon = T$ and after renaming positions, i.e. the initial ($q_0 = q'$) and the final ($q_N = q''$) one, (58) can be written as (for details see e.g. [20],[21])

$$K(q'', T \mid q') = \lim_{N\to\infty} B_N(T) \int dq_1 ... dq_{N-1} \exp\{\frac{i}{\hbar}\sum_{i=0}^{N-1} \varepsilon[\frac{m}{2}(\frac{q_{i+1} - q_i}{\varepsilon})^2 - V(q_i)]\}$$

$$=: \int_{q',t'}^{q'',t''} \mathcal{D}q(t)\, e^{\frac{i}{\hbar}\int_T dt[\frac{1}{2}\dot{q}^2 - V(q)]} = \int_{q',t'}^{q'',t''} \mathcal{D}q(t)\, e^{\frac{i}{\hbar}S[q(t),T]}. \tag{60}$$

The factor $B(T)$, which determines the "measure" of the path integral, will be adjusted to make the integral finite and suitably normalized as $N \to \infty$. Thus the path integral is defined as a limiting process of a discrete lattice calculation. This "derivation" suffers from certain problems (for details see [21]). (i) The "$H(p, q)$-symbol" [21] in general depends on the ordering-prescription for the operators $\hat{q}$ and $\hat{p}$ (this is no problem for Hamiltonians of the form

(55)) and can differ from the classical Hamiltonian and therefore the classical action, by terms proportional to $\hbar$. This problem exists already in the operator formalism where the derivation starts. (ii) There also exists a time ordering ambiguity for the momentum integrations, which are already carried out in (60). For a different prescription of the integrations one would have to change the ordering prescription for the operators, i.e. choose a different H-symbol, to get the same amplitude. Thus the ordering ambiguities when passing from the classical Hamiltonian H to the operator $\mathcal{H}$ do not disappear. (iii) The existence of the (complex) measure for the functional integration. The problem can be skipped by evaluating the Euclidean path integral and analytic continuation, if it exists, to the (real-time) Minkowski space. But the existence of the measure is still in question.

Nevertheless we will see that the path integral is a comfortable tool for our calculations. The theories which we consider here do not suffer from the operator ordering problem in the derivation above and thus are free from the evaluation prescription problems on the lattice.

The q- representation of the Fourier transformed kernel, which depends on the period T instead of the energy E, gives the (outgoing) *Green function* (also called propagator)[10]

$$G(q''|q', E) = \langle q''| \frac{1}{(\mathcal{H} - E - i\epsilon)} |q'.\rangle \tag{61}$$

Thus the singularities (for $\epsilon \to 0$) are poles of the Green function. The kernel K is related to the Green function G by the inverse Fourier transformation of (61):

$$K(q'', T|q') = \int_{-\infty}^{\infty} \frac{dE}{2\pi i}\, e^{-\frac{i}{\hbar}ET} G(q''|q', E).$$

Spectral representation and energy levels. In general the spectrum of the Hamiltonian will consist of a discrete part (bound states) and a continuous part (scattering states), i.e

$$\mathcal{H}\,|n\rangle = E_n\,|n\rangle \quad n = 0, 1, 2 \ldots N \tag{62}$$

$$\mathcal{H}\,|p\rangle = E(p)\,|p\rangle \quad p \in \mathbb{B}, \tag{63}$$

where the "domains" N and $\mathbb{B}$ will depend on the considered system, usually is $\mathbb{B} = \mathbb{R}$. The states in (62, 63) are orthonormal and fulfill the completeness relation

$$\sum_n^N |n\rangle\,\langle n| + \int_{\mathbb{B}} dp\,|p\rangle\,\langle p| = \mathbf{1} \tag{64}$$

with the wave functions

$$u_n(x) := \langle q|n\rangle \quad \text{and} \quad \varphi(p, x) := \langle q|p\rangle\,.$$

By inserting the completeness relation (64) into the expressions for the kernel (58) and the Green function (61) one obtains a sum of the discrete and the continuous part:

$$K(q'', T|q') = \sum_n^N u_n(q'')u_n^*(q')\, e^{-\frac{i}{\hbar}E_nT}\theta(T) + \int_{\mathbb{B}} dp\varphi(p, q'')\varphi^*(p, q')\, e^{-\frac{i}{\hbar}E(p)T}\theta(T) \tag{65}$$

$$G(q''|q', E) = \sum_n^N \frac{u_n(q'')u_n^*(q')}{E(p) - E - i\epsilon} + \int_{\mathbb{B}} dp \frac{\varphi(p, q'')\varphi^*(p, q')}{E(p) - E - i\epsilon}. \tag{66}$$

[10]The names for $\mathcal{K}$ and $\mathcal{G}$ are author dependent. We use the notion kernel for $\mathcal{K}$ and Green function for the Fourier transformed version $\mathcal{G}$.

As one can see, for the discrete spectrum we identify the poles with the bound state energies and the residues with the bound state wave functions. For the continuous spectrum of $\mathcal{H}$ the Green function G has a branch cut. In the following we will use a more symbolic notation and write a single infinite sum for both, the discrete and the continuous spectrum.

The generalization of the above formulas to more DOF is straightforward. The coordinate q, $q(t)$ simply becomes q_m, $q_m(t)$. For the Schrödinger wave functions (59) this means

$$\psi(q_1, \ldots, q_N, t) = \langle q_1, \ldots, q_N | \psi(t) \rangle \,. \tag{67}$$

In field theory, which can formally be obtained by the limit $\lim_{N\to\infty} \{q_1(t), \ldots, q_N(t)\} \to \phi(x,t)$, the wave function (67) becomes a functional of the field ϕ ("state functional"), an exceedingly complicated object, especially for nontrivial field configurations such as solitons. Therefore we will not calculate the kernel K (U) or the Green function G, but their trace, so one never has to construct any state functionals. These methods are based on the work of Dashen et.al [23].

Spectral function and Feynman-Kac-formula. We are interested in the energy-spectrum of a system, especially the lowest energy in the presence of a nontrivial classical solution, to calculate the correction to the classical (kink) masses. In the case of static classical solutions this is done by investigating the trace of the time-evolution operator U resp. the kernel K (note that for $T > 0$, which we assume in the following, $U = K$). If in (65) we set $q'' = q' := q_0$, i.e. we consider *closed paths*, and integrate over all possible initial conditions q_0, one obtains for the trace[11] , using (65)

$$\text{Tr } e^{-\frac{i}{\hbar}\mathcal{H}T} = \int dq_0 K(q_0, T \mid q_0) = \sum_n e^{-\frac{i}{\hbar}E_n T} \int dq_0 \mid u_n(q_0) \mid^2 . \tag{68}$$

This is the *spectral function* of the theory, defined by $\mathcal{H}$. Analytic continuation, which is related to the Euclidean path integral, to complex variable $\tau = \frac{i}{\hbar}t$ and taking the limit $\tau \to \infty$ picks out the ground state energy of the sum in (68), thus

$$\lim_{\tau\to\infty} e^{E_0\tau} \int dq_0 K(q_0, -i\hbar\tau \mid) = k,$$

where k is the multiplicity of the ground state, i.e. the degree of degeneration. For a non-degenerated ground state ($k = 1$) one obtains for the limit of the logarithm

$$E_0 = \lim_{\tau\to\infty} \frac{1}{\tau} \ln \int dx K(q_0, -i\hbar\tau \mid q_0) = \lim_{\tau\to\infty} \frac{-1}{\tau} \ln \text{Tr } e^{-\mathcal{H}\tau}.$$

This is the Feynman-Kac-formula and it allows to calculate the ground state energy without detailed knowledge of K. Nevertheless we will directly calculate the spectral function (68), in a perturbative calculation, and read off the energy spectrum.

The trace in (68) can be written as a path integral (60) for closed paths with an additional integration over the initial=final position $q_0 = q_N$:

$$K(T) := \text{Tr } e^{-\frac{i}{\hbar}\mathcal{H}T} = \int dq_0 K(q_0, T \mid q_0) = \int dq_0 \int_{q_0,t'}^{q_0,t''} \mathcal{D}\, q(t) e^{\frac{i}{\hbar}S[q(t),T]}. \tag{69}$$

To evaluate the trace we use approximation techniques for the path integral and obtain in this way approximate energy levels of the system.

[11] We use the symbolic notation for the discrete and continuous spectrum, as mentioned above.

3.1.2 Stationary phase approximation (SPA) and perturbation theory

In general it is not possible to exactly evaluate the kernel K. Thus one has to use some approximation techniques (perturbation theory) to calculate K. As can be seen from the path integral representation of K, one has to deal with functional generalization integrals of the form:

$$F(\beta) = \int dx\, e^{i\beta f(x)}. \tag{70}$$

This is also true in field theory. The idea is to get the dominant contribution of these integrals for $\beta \to \infty$. In quantum theory the perturbation parameter is $\beta = \frac{1}{\hbar}$ or by a rescaling of the fields the dimensionless parameter $\frac{1}{\lambda\hbar}$, respectively (see section 2.5). The limit $\beta \to \infty$ either corresponds to limit $\hbar \to 0$, which means that the action $S \gg \hbar$ (*semi-classical expansion*) or $\lambda \to 0$ (weak coupling), which is the situation in *standard perturbation theory*, and there is no need for the action to be $S \gg \hbar$. In this limit the integrand oscillates very fast and by the lemma of Riemann-Lebesgue the integral vanishes. The leading contribution comes from the stationary region of the phase $f(x)$ (this corresponds to $\delta S = 0$), i.e. from those values of x near x_0, with $f'(x_0) = 0$. In a first approximation we expand $f(x)$ around x_0 and neglect terms of $O((x - x_0))^3$. This gives for (70)

$$F(\beta) = e^{i\beta f(x_0)} \sqrt{\frac{2\pi i}{\beta f''(x_0)}} + O(\frac{1}{\beta}), \tag{71}$$

where we have assumed that $f''(t_0) \neq 0$. The $f''(t_0) = 0$ - case needs an extra examination[12]. That regions of x for which $f'(x) \neq 0$ only give contributions of $O(\frac{1}{\beta})$ can be seen as follows: Let $f'(x) \neq 0$ for $a < x < b$, then we can change to the variable $z = f(x)$ in (70). Thus

$$F_{ab} = \int_a^b dx\, e^{i\beta f(x)} = \frac{1}{i\beta} \left[\frac{1}{f'} e^{i\beta z} \Big|_{f(a)}^{f(b)} - \int_{f(a)}^{f(b)} dz\, e^{i\beta z} \frac{d}{dz} (f')^{-1} \right].$$

Hence F_{ab} goes to zero like $\frac{1}{\beta}$ as $\beta \to \infty$, where regions having $f' = 0$ are of order $\frac{1}{\sqrt{\beta}}$ and therefore dominate in this limit. To illustrate the difference and the connection between standard perturbation theory, respectively, i.e. perturbation theory around the vacuum, and perturbation theory around a nontrivial stationary point (soliton) we examine exponents of the form

$$f(x) = x^2 + v(x;\lambda) \text{ with } v(x;\lambda) = \frac{1}{\lambda} v(\lambda x) \text{ and } v = O(\geq x^3).$$

$v(x;\lambda)$ is the "interaction-term" (for example $v(x;\lambda) = -\lambda x^4$). We assume that $f(x)$ has two stationary points. The trivial one ("vacuum"), $x = x_V = 0$ with $f(x_V) = 0$ being the absolute minimum and a nontrivial one, $x_s \neq 0$ and $x_s = O(\frac{1}{\sqrt{\lambda}})$ ("soliton"), for which $f(x_s)$ is large relatively to the scale β ("$\frac{1}{\hbar}$"). Expanding the exponent around the trivial stationary point $x = 0$ ("the vacuum") one obtains ($y = x - x_V$)

$$F \sim \int dy\, e^{i\beta y^2} \sum_{n=0}^{\infty} \frac{(i\lambda\beta)^n}{n!} \left(P(x)\right)^n, \tag{72}$$

[12]This case is related to the *zero mode* problem.

where $P(x)$ is a polynomial which one gets by expanding the "interaction" $v(x;\lambda)$ around x_V and factoring out the coupling λ. This is a perturbative expansion in the coupling λ which is reasonable in the weak-coupling regime ($\frac{\beta}{\lambda} \ll 1$) for which also the integrand is oscillating very fast, although the "action" $f(x)$ for the vacuum is zero as mentioned above, and thus the stationary phase approximation is applicable. The symbol $\sim$ indicates that this perturbative expansion is in general only an asymptotic series and not a convergent one [20]. Expanding now the "action" around the nontrivial stationary point x_s one obtains ($y = x - x_s$)

$$F \sim e^{i\beta f(x_s)} \int_0^\infty dy\, e^{i\beta \frac{1}{2} f''(x_s) y^2} \sum_{n=0}^{\infty} \frac{i^n}{n!} \left(\beta \tilde{P}(y)\right)^n,$$

where $\tilde{P}(y)$ is a polynomial of $O(\geq y^3)$, obtained by expanding the "interaction" v around x_s. Again this perturbative expansion is in general only an asymptotic series. This expansion is reasonable if $\beta\tilde{P}$ is small, i.e. the contributions of the deviations y of the nontrivial classical solution x_s to the action are small relative to the scale β ("$\frac{1}{\hbar}$"). That the integrand oscillates very fast is due to the nontrivial solution x_s for which the action $f(x_s)$ is large relative to the scale β ("$\frac{1}{\hbar}$"), i.e that $\beta f(x_s) \gg 1 \Leftrightarrow \beta \to \infty$.

We can collect the ingredients of these perturbation expansion as follows

1. *Standard perturbation theory* (= weak coupling). The action is expanded around the trivial classical solution $x_v = O(\lambda^0)$, the action $f(x_v)$ and the energy of this solution is zero $\Rightarrow$ x_v is the vacuum. For a weak coupling λ (dimensionless $\frac{\lambda}{\beta} \ll 1 \Leftrightarrow \beta \to \infty$) the integrand is oscillating very fast $\Rightarrow$ both, stationary phase approximation (expansion around $f(x_v)$) as higher order perturbative expansion (= expansion in λ) is reasonable. The perturbative expansion is an expansion around the free "field modes" (the Gaussian integration in (72) respects only the quadratic action ($\lambda = 0$) which gives the free Feynman propagator in field theory).

2. *Non-trivial perturbation theory* (= "large action"). The action is expanded around a nontrivial classical solution $x_s = O(\frac{1}{\sqrt{\lambda}}) \Rightarrow$ *non perturbative*, the action $f(x_s)$ is large relative to the scale $\beta \equiv \frac{1}{\hbar} \Rightarrow f(x_s)\beta \gg 1 \Leftrightarrow \beta \to \infty$, therefore the stationary phase approximation is reasonable. The contributions of the "paths" nearby the non-trivial classical solution to the action are small, i.e with $f(x) = f(x_s) + f''(x_s)y^2 + \Delta f(y)$ is $\beta \Delta f(y) \equiv \frac{\Delta f(y)}{\hbar} \ll 1$, therefore the "semi-classical perturbative expansion" (= expansion in $\hbar$) is reasonable, even in a strong coupling regime as long as this does not violate the above requirements. The asymptotic states are not free fields as in standard perturbation theory.

3. *SPA, semi-classical approximation.* For the quadratic term $f^{(2)}(y) := f''(x_s)y^2$ being the dominant correction, it should be of order $\frac{1}{\beta} \equiv \hbar$, i.e. $\beta f^{(2)} = O(1)$. In this case (71) gives the leading correction and is called stationary phase approximation (SPA) or semi-classical approximation[13].

We will be mostly interested in the second case, where the nontrivial stationary points of the action will be the solitonic solutions of section 2.

[13] In the literature also the notion WKB method is used, but we want to reserve this notion for special cases of the SPA.

In the case of multiple integrals where f depends on N variables q_i the expansion of f around an extremum at $\vec{q} = \vec{a}$ writes as $(y_i = q_i - a_i)$

$$f(\vec{q}) = f(\vec{a}) + \frac{1}{2} y_i A_{ij} y_j + O(y^3) \ , \ A_{ij} = \frac{\partial^2 f}{\partial q_i \partial q_j}(\vec{a}) \tag{73}$$

and the integral gives

$$\int dq_1 ... dq_N e^{i\beta f(\vec{q})} = e^{i\beta f(\vec{a})} (2\pi i)^{N/2} (\frac{1}{\beta})^{N/2} \frac{1}{\sqrt{det \mathrm{A}}} + O(\frac{1}{\beta})^N.$$

Here again we have assumed that no eigen-value of the matrix A is zero. Closely related to the method of stationary phase is Laplace's method for integrands of the form $\exp(-\beta g(t))$, where $g(t)$ is bounded from below. Laplace's method takes the place of the SPA in the Euclidean path integral formulation.

Stationary phase approximation for the path integral. The "phase" in the path integral is the action $S[q]$. Thus an approximation around the stationary phase means an approximation around classical paths $q_{cl}(t)$ for which the action is stationary,

$$\delta S|_{q_{cl}} = 0 \text{ with } q_{cl}(t') = q' \text{ and } q_{cl}(t'') = q'',$$

and the values q', q'' are the initial and the final position for the kernel (60). For simplicity we consider $D = 1$ quantum mechanics with the particle-action

$$S[q] = \int_0^T dt [\frac{1}{2}\dot{q}^2 - V(q)].$$

Expanding this action around the classical path $q_{cl}(t)$ (*shifting method*) gives

$$\begin{aligned} & S[q] = S[q_{cl} + \eta] \\ & = S(q_{cl}) + \dot{q}\eta|_0^T + \left[\frac{1}{2} \int_0^T dt\eta \left(-\partial_t^2 - V''|_{q_{cl}}(t) \right) \eta + \eta\dot{\eta}|_0^T \right] + \sum_{k=3}^{N} \int_0^T dt \frac{1}{k!} V^{(k)}|_{q_{cl}}(t)(\eta)^k \\ & =: S(q_{cl}) + \delta S|_{q_{cl}} + \frac{1}{2}\delta^2 S|_{q_{cl}} + \sum_{k=3}^{N} \frac{1}{k!} \delta^k S|_{q_{cl}} = S(q_{cl}) + \delta S|_{q_{cl}} + \frac{1}{2}\delta^2 S|_{q_{cl}} + \Delta S \end{aligned}$$

The surface terms $\frac{1}{2}\eta\dot{\eta}|_0^T$ and $\dot{q}\eta|_0^T$ vanish if the classical path connects the initial and final position q', q'' in the kernel (60), since in this case is $\eta(0) = 0 = \eta(T)$. This is not always true[14]. The classical action $S(q_{cl})$ is the action evaluated for the classical path q_{cl} and thus a normal function of T. Therefore we will often write $S_{cl}(T)$ for this term. The first variation δS is of course zero (up to possible boundary terms) for the classical path. The operator $\mathcal{O} = -\partial_t^2 - V''(t)$ is the analogue of the matrix A_{ij} of (73). The term ΔS_I gives higher order corrections. Thus we approximate the path integral of the kernel (60) as follows:

$$K(q'', T \mid q') = \int_{q',t'}^{q'',t''} \mathcal{D}q(t) e^{\frac{i}{\hbar} S[q,T]} \approx e^{\frac{i}{\hbar} S_{cl}(T)} \int_{0,t'}^{0,t''} \mathcal{D}\eta(t) \ e^{\frac{i}{\hbar} \frac{1}{2} \int_T dt \eta \mathcal{O} \eta} \tag{74}$$

$$= e^{\frac{i}{\hbar} S_{cl}(T)} B'(T) \frac{1}{\sqrt{det \mathcal{O}}}. \tag{75}$$

[14] When using functional derivatives one has to be very careful if they really exist, this is only true if such boundary terms do not occur. Otherwise one loses automatically these boundary terms. This is also different from a variation principle, which is defined by fixing the variation at the endpoints and corresponds to the boundary conditions for the e.o.m. In standard perturbation theory one does not have to care about surface terms, since one considers an unbounded time interval $(-\infty, \infty)$.

The pre-factors and the measure constant $B(T)$ are absorbed in the new constant $B'(T)$. This approximation is also called *semi-classical approximation* since the sum over all paths is approximated by the sum over the classical path $q_{cl}(t)$ and paths in its neighborhood. The quantum effects (corrections) are included in the factor

$$B'(T)\frac{1}{\sqrt{det\mathcal{O}}}$$

When calculating the determinant $det\mathcal{O}$ one has to respect boundary conditions, in this case the homogeneous one, $\eta(0) = \eta(T) = 0$. In more general cases, i.e. if the classical path does not exactly connect the positions q', q'', one has to choose the boundary conditions in a way, so that the set of fluctuations $\{\eta\}$ form a linear space, in which the operator $\mathcal{O}$ acts. This is the advantage of the "shifting method", that the "path integration domain" (PID) becomes a linear space. If no classical solution is available, the PID can be turned into a linear space by the following variable transformation in the path integral [19]

$$\hat{q}(t) := q(t) - [\frac{q''(t-t') + q'(t''-t)}{(t''-t')}].$$

Again for this approximation we have assumed that no eigen-value of the operator $\mathcal{O}$ vanishes. Vanishing eigenvalues will lead us to the *zero-mode problem* and spoil the conditions for the validity of the SPA.

If the action has several stationary points then each gives an additional separate contribution, provided the paths which make the action stationary are not too "close" to each other, since otherwise the condition for the SPA, that paths near the classical one, the fluctuations η, give only small contributions to the action ($\frac{\Delta S}{\hbar} \ll 1$, see point 2. above), is not fulfilled. A characteristic length for the validity of the SPA is heuristically obtained as follows:

If the quadratic term dominates, then we have

$$\frac{\delta^2 S[\eta]}{\hbar} = O(1) \text{ as } \hbar \to 0$$

This means that η is on the order of $\sqrt{\hbar}$, i.e.

$$\eta = O(\sqrt{\hbar}) \text{ as } \hbar \to 0$$

This is a relevant length for the SPA. Assuming that there exist several classical paths $q_\alpha, q_\beta, ...$, their distance must be larger than the characteristic length. Well, the distance is measured by the action. Let be $q = q_\alpha + \eta = q_\beta$, then an expansion of the action gives

$$S[q_\alpha + \eta] = S(q_\alpha) + \delta^2 S_\alpha + \Delta S_\alpha = S(q_\beta).$$

In order not to spoil the conditions of the SPA, for the difference in the actions of two classical paths must satisfy:

$$S(q_\beta) - S(q_\alpha) = \delta^2 S_\alpha + \Delta S_\alpha \gg \hbar \tag{76}$$

$$\Rightarrow \int dt(q_\alpha - q_\beta)\frac{\delta^2 S}{\delta q^2}|_{q_\alpha}(q_\alpha - q_\beta) \gg \hbar. \tag{77}$$

Otherwise the paths are near a focal point which also leads to the zero-mode problem. Especially in the presence of continuous symmetries these problems occur (see section 2.6), since

an infinitesimal transformation of a classical path q_α can give a continuous set of neighboring paths with the same action.

Higher order corrections. The higher order corrections one gets from perturbation theory using the "rest" of the action. The interaction which is treated perturbative reads

$$\Delta S =: S[q] - S_{SPA} = \sum_{k=3}^{N} \frac{1}{k!} \delta^k S = \sum_{k=3}^{N} \int_0^T dt \frac{1}{k!} V^{(k)}|_{q_{cl}}(t)(\eta)^k \tag{78}$$

With this definition the path integral can be written as

$$\begin{aligned} K(q'', T \mid q') &= \int_{q',0}^{q'',T} \mathcal{D}q(t)\; e^{\frac{i}{\hbar} S_{SPA}}\; e^{\frac{i}{\hbar}\Delta S} \\ &= \int_{q_a,0}^{q_b,T} \mathcal{D}q(t)\; e^{\frac{i}{\hbar} S_{SPA}} \sum_{m=0}^{\infty} \frac{1}{m!} \left(\frac{i\Delta S}{\hbar}\right)^m . \end{aligned}$$

Thus for further perturbation theory $\Delta S < \hbar$ must be valid. This leads to generalized Feynman graphs, but of course more complicated since the "couplings" $V^{(k)}|q_{cl}(t)$ are time dependent. In field theory this was done in [23].

3.1.3 One exactly solvable problem, the harmonic oscillator

We now calculate the spectral function (69), i.e. the trace of the kernel, for the harmonic oscillator. Although this is a well known and trivial system we treat it in some more detail. Especially we are interested in the explicit expression of the measure, since we will end up with the harmonic oscillator every time. The Lagrangian and the e.o.m. (obtained by a variation principle, i.e. an extremum of the action, with vanishing variation of the endpoints $\delta q(0) = \delta q(T) = 0$) are given by:

$$\begin{aligned} L &= \tfrac{1}{2}(\dot{q}^2 - \omega^2 q^2) \\ \ddot{q} + \omega^2 q &= 0 \end{aligned}$$

The classical solution with the *closed path BC* $q(0) = q(T) = q_0$ for the trace, and the associated action-function $S_{cl}(q_0, T)$ for these trajectories are given by [15](assuming $\omega T \neq n\pi, n \in \mathbb{N}$)

$$q_{cl}(t) = q_0 \left(\cos\omega t + \tfrac{2\sin^2\frac{\omega T}{2}}{\sin\omega T} \sin\omega t\right) \tag{79}$$

$$S^{cl}(q_0, T) = \tfrac{1}{2}\int_0^T dt(\dot{q}_{cl}^2 - \omega^2 q_{cl}^2) = -2\omega q_0^2 \tfrac{\sin^2(\frac{\omega T}{2})}{\sin\omega T} \tag{80}$$

Here and and in the following we use a mode-expansion method to evaluate the path integral, rather than a lattice calculation. For this we expand the exponent in (69), i.e the action

$$S[q] = \frac{1}{2}\int_0^T dt[\dot{q}^2(t) - \omega^2 q^2(t)] \tag{81}$$

[15]For our purposes this singular situation can always be avoided, since we can choose the period T arbitrary.

around the classical solution (79), i.e. $q(t) = q_{cl}(t) + \eta(t)$. The expansion terminates after the second order since (81) is quadratic in $q(t)$. So the following is exact and so is the semi-classical calculation[16]:

$$S[q] = S[q_{cl} + \eta] = S^{cl}(q_0, T) + \tfrac{1}{2}\int_0^T dt\eta(t)\{-\partial_t^2 - \omega^2\}\eta(t) \tag{82}$$

$$\text{with } \eta(0) = 0 = \eta(T) \ldots \text{closed path BC} \tag{83}$$

Because of the closed path BC the fluctuations fulfill Dirichlet boundary conditions. The advantage of the shifting of the path integration over the paths $q(t)$ to a path integration over fluctuations $\eta(t)$ around the classical path is that the set of fluctuations $\eta(t)$ form a linear space, as long the boundary conditions are linear relations, in contrast to the class of paths from $q(0) = q_0$ to $q(T) = q_T$ (the sum of two paths goes from $2q_0$ to $2q_T$). This is necessary for the functional integration of the exponent (83), since this is done by diagonalization of the differential operator $\mathcal{O} = -\partial_t^2 - \omega^2$ *in the path integration domain PID.* Thus the path integration domain *must be a linear space.*

The diagonalization is done by solving the homogeneous (BC) eigenvalue problem for the differential operator $\mathcal{O}$. $\mathcal{O}$ is a Schrödinger-like operator and thus has an ordered spectrum ($\lambda_1 < \lambda_2 \ldots$) and a complete orthonormal set (in the sense of the space $L^2(\mathbb{R})$) of eigenfunctions. The homogeneous (BC) eigenvalue problem reads

$$(-\partial_t^2 - \omega^2)\psi_n = \epsilon_n\psi_n \ , \ \psi_n(0) = \psi_n(T) = 0 \tag{84}$$

The solution is easily obtained:

$$\epsilon_n = k_n^2 - \omega^2 \qquad k_n = \frac{n\pi}{T} \ n = 1, 2, \ldots \tag{85}$$

$$\psi_n(t) = \theta(T-t)\sqrt{\tfrac{2}{T}}\sin k_n t \quad \int_0^T dt\psi_n\psi_{n'} = \delta_{n,n'} \tag{86}$$

The set of fluctuations $PID = \{\eta \in \mathcal{C}[0,T] \mid \eta(0) = \eta(T) = 0\}$ is "larger" than the space $\{L^2([0,T],$ homogenous Bc$)\}$ in which the set $\{\psi_n \mid n = 1, 2, \ldots\}$ forms a basis, since one has to consider *all* fluctuations η, even those which are not square-integrable on the interval $[0,T]$. Anyhow, we expand each deviation $\eta(t)$ according to the basis $\{\psi_n(t)\}$. The correctness of the final result seems to say that the set of fluctuations not included in this expansion is of "measure" zero. The reason for this is that the paths far away from the stationary point interfere destructively (see also the comments on the characteristic length of the SPA, which is exact here). To have well defined expressions we make a finite expansion, which is called a mode regularization:

$$\eta(t) = \sum_{n=1}^{N} a_n\psi_n(t) \Longrightarrow \eta(0) = \eta(T) = 0. \tag{87}$$

As one can see, because of the implementation of the (linear) BC on the individual modes ψ_n the full fluctuation field η automatically fulfills the required BC. The coefficients are completely free since the information on the BC is encoded in the eigen-values ϵ_n. By this expansion according to the fixed basis $\{\psi_n\}$ a variation in the function $\eta(t)$ means a variation in its

[16]Surface terms again vanishes because of the closed path BC

coefficients a_n. Thus a path integration over $\eta(t)$ means an integration over the coefficients a_n. For the action (82)we get

$$S[q] = S[q_{cl} + y] = S^{cl}(q_0, T) + \frac{1}{2}\sum_{n=1}^{N} \epsilon_n a_n^2.$$

For the trace of the time-evolution-operator (=Kernel for $T > 0$) one obtains

$$\mathrm{Tr} e^{-\frac{i}{\hbar}\mathcal{H}T} = \int dq_0 \int_{q_0,0}^{q_0,T} \mathcal{D}q \; e^{\frac{i}{\hbar}S[q]} = \int dq_0 e^{\frac{i}{\hbar}S^{cl}(q_0,T)} \int_{0,0}^{0,T} \mathcal{D}\eta(a_n) \; e^{\frac{i}{\hbar}\sum_n \epsilon_n a_n^2}.$$

The measure $\mathcal{D}\eta$ is given as

$$\mathcal{D}\eta := B_N(T) \prod_{n=1}^{N} da_n$$

where $B(T)$ is an appropriate normalization constant which will be *defined* below. It is important to note that the mode-expansion-evaluation is independent of the lattice calculation although there exists a one-to-one correspondence at least for bosons, between them. But the expansion (87) is *not* an ordinary variable transformation from the lattice points $q(t_i) = q_i$ to the mode coefficients a_n [22]. Therefore the measure $B(T)$ must be defined by proper normalization conditions independent of the measure given on the lattice; it cannot be obtained by the Jacobian of a transformation $q_i \to a_n$. The trace is now

$$\mathrm{Tr} e^{-\frac{i}{\hbar}\mathcal{H}T} = \int dq_0 \; e^{\frac{i}{\hbar}S^{cl}(q_0,T)} B_N(T) \prod_{n=1}^{N} \int da_n \; e^{\frac{i}{\hbar}\epsilon_n a_n^2} = \int dq_0 e^{\frac{i}{\hbar}S^{cl}(q_0,T)} B_N(T) (i\pi\hbar)^{\frac{N}{2}} \prod_{n=1}^{N} \frac{1}{\sqrt{\epsilon_n}}.$$

The *finite* product of eigen-values ϵ_n of the operator $-\partial_t^2 - \omega^2$ is the *regularized* functional determinant of this operator in the space spanned by $\{\psi_n\}$. For the eigenvalues of the harmonic oscillator (86) the product can written down in closed form

$$B_N(T)(i\pi\hbar)^{\frac{N}{2}} \prod_{n=1}^{N} \frac{1}{\sqrt{\epsilon_n}} = \frac{B_N(T)(i\pi\hbar)^{\frac{N}{2}}}{\sqrt{det\mathcal{O}_{BC}}} = B_N(T)(i\pi\hbar)^{\frac{N}{2}} \prod_{n=1}^{N} \left(\frac{n^2\pi^2}{T^2} - \omega^2\right)^{-\frac{1}{2}}$$
$$= B_N(T) \left(\frac{i\hbar}{\pi}\right)^{\frac{N}{2}} \frac{T^N}{N!} \prod_{n=1}^{N} \left(1 - \frac{\omega^2 T^2}{n^2\pi^2}\right)^{-\frac{1}{2}} \longrightarrow \lim_{N\to\infty} \left[B_N(T) \left(\frac{i\hbar}{\pi}\right)^{\frac{N}{2}} \frac{T^N}{N!}\right] \sqrt{\frac{\omega T}{\sin \omega T}}$$

For the limit $N \to \infty$ we have used that both factors exist by themselves. For the second product, involving the dynamics through ω, this is a standard formula [7]. For the first product we have assumed that the measure $B_N(T)$ is chosen in such a way that the product also exists, and in the final normalization we will see that this is true. Finally one obtains for the trace

$$\mathrm{Tr} e^{-\frac{i}{\hbar}\mathcal{H}T} = \lim_{N\to\infty} \left[B_N(T) \left(\frac{i\hbar}{\pi}\right)^{\frac{N}{2}} \frac{T^N}{N!}\right] \int dq_0 e^{-i\frac{2\omega q_0^2}{\sin \omega T} \sin^2(\omega T/2)} \sqrt{\frac{\omega T}{\sin \omega T}}$$
$$= \lim_{N\to\infty} \left[B_N(T) \left(\frac{i\hbar}{\pi}\right)^{\frac{N}{2}} \frac{T^N}{N!}\right] \sqrt{2\pi i T} \frac{1}{2i \sin(\omega T/2)} \quad .$$

Writing the sine as exponentials and using the formula for the geometric series one obtains

$$\mathrm{Tr}e^{-\frac{i}{\hbar}\mathcal{H}T} = \lim_{N\to\infty}\left[B_N(T)\left(\frac{i\hbar}{\pi}\right)^{\frac{N}{2}}\frac{T^N}{N!}\right]\sqrt{2\pi i}\sum_{\nu=0}^{\infty}e^{-i\omega(\nu+\frac{1}{2})T}. \tag{88}$$

Finally we have to fix the measure by a normalization condition. For obvious reasons we choose

$$\lim_{N\to\infty}\left[B_N(T)\left(\frac{i\hbar}{\pi}\right)^{\frac{N}{2}}\frac{T^N}{N!}\right]\sqrt{2\pi i} = 1 \;\Rightarrow\; B_N(T) = \frac{1}{\sqrt{2\pi i}}\left(\frac{\pi}{i\hbar}\right)^{\frac{N}{2}}\frac{N!}{T^N}. \tag{89}$$

Here we can see that the suggestive notation $B(T)$ for the measure constant is justified, since it does not depend on the dynamics, i.e. on ω, and is thus purely kinetic and for all harmonic oscillators (different ω's) the same. With this normalization we can read off the energy spectrum of the harmonic oscillator from its spectral function (88) as follows

$$E_\nu = \hbar\omega(\nu+\frac{1}{2})\;,\;\nu = 0,1,\ldots$$

This is the well known spectrum of the harmonic oscillator. The normalization-condition for the measure (89) is unique up to factors of the form e^{-icT}, where c is a constant, since this would shift the energy spectrum by the constant c which corresponds to the freedom of choosing the ground state energy. Up to this freedom the normalization is unique and corresponds to the wave-function normalization, as can be seen from (68). As one can also see the measure constant $B(T)$ does not exist by itself, but this is a well known situation which also occurs in Wiener integrals and the reason for this is that the exponential of the velocity term in the action is part of the functional measure [30].

We have treated these fundamental, perhaps trivial affairs like $1D$ quantum mechanics and the harmonic oscillator in such detail because the more complicated (field) systems, which we will consider in the following, will always be traced back to these "simple" foundations.

3.1.4 Field theory

In field theory one changes from functions $q(t)$, depending on one parameter, to functions $\phi(\vec{x},t)$, depending on the parameters $\{\vec{x},t\}$. Or one can say that instead of the correlation of one number $q(t)$ for each t one has an infinite set of numbers $\phi(\vec{x},t)$ for each t. This "view" of quantum field theory is sometimes very helpful (see e.g. the lattice resp. the mode-expansion formulation of the path integral). But of course relativistic field theory is more than the formal limit to an infinite number of degrees of freedom (DOF).

Field representation. The formulas of the above sections are straightforward to generalize to field theory. The analogue of the coordinate representation (56), defined by eigen-states of the position operators in the Schrödinger picture, is the field representation, defined by the field operator in the Schrödinger picture (i.e. at a fixed time):

$$\hat{\phi}(\vec{x})\,|\phi(\vec{x})\rangle = \phi(\vec{x})\,|\phi(\vec{x})\rangle\,,$$

or alternatively one cane use (time-dependent) Heisenberg operators

$$\hat{\phi}_H(\vec{x},t)\,|\phi(\vec{x}),t\rangle_H = \phi(x,t)\,|\phi(\vec{x}),t\rangle_H$$

These two pictures are connected as usually by

$$|\phi(\vec{x}),t\rangle_H = e^{\frac{i}{\hbar}\mathcal{H}t}\,|\phi(\vec{x})\rangle\,.$$

The kernel (propagator) is now the amplitude that the system evolves from a field configuration $\phi_a(\vec{x})$ at $t=t'$ to a field configuration $\phi_b(\vec{x})$ at a (later) time $t=t''$ ($T=t''-t'$), and reads:

$$K(\phi_b(\vec{x}),T|\phi_a(\vec{x})) = \langle\phi_a(\vec{x})|\,e^{-\frac{i}{\hbar}\mathcal{H}T}\,|\phi_b(\vec{x})\rangle = \int_{\phi_a(\vec{x}),t'}^{\phi_b(\vec{x}),t''} \mathcal{D}\phi(\vec{x},t)e^{\frac{i}{\hbar}S[\phi,T]} \tag{90}$$

$$=_H \langle\phi(\vec{x}),t''|\phi(\vec{x}),t'\rangle_H \tag{91}$$

where the action is given by

$$S[\phi,T] = \int_0^T dt \int\limits_{space} dx\mathcal{L}(\phi(\vec{x},t))$$

The field representation is useful only for general (formal) considerations. For a lattice calculation the path integral is now defined on a space-time lattice. To show that there exists a unique Lorentz-invariant limit on the space-time lattice is of course a nontrivial problem, since there are a lot of possible kinds of lattice-structures in $D>1$ dimensions. We will again use the mode-expansion method and assume that the functional integral exists uniquely.

For the following paragraph we use $\vec{x}=x$.

Spectral function. For the trace formula (68) one needs one more integration over the initial=final field configuration. This is again a functional integral in the case of fields. For the trace we again evaluate the kernel for closed paths. In field theory this means $\phi(x,t')=\phi(x,t'')=:\phi_a(x)$, and integrate in addition over the initial=final field configuration $\phi_a(x)$. As before we insert a complete set of energy eigen-states (symbolic notation) in (90). So we get:

$$\int \mathcal{D}\phi_a(x)\sum_n \langle\phi_a(x)|n\rangle\,\langle n|\phi_a(x)\rangle\;\; e^{-\frac{i}{\hbar}E_nT} \tag{92}$$

$$= \sum_n \langle n|\left(\int \mathcal{D}\phi_a(x)\,|\phi_a(x)\rangle\,\langle\phi_a(x)|\right)|n\rangle\, e^{-\frac{i}{\hbar}E_nT} \tag{93}$$

$$= \sum_n e^{-\frac{i}{\hbar}E_nT} = \mathrm{Tr}\; e^{-\frac{i}{\hbar}\mathcal{H}}. \tag{94}$$

Here we have used the completeness relation

$$\int \mathcal{D}\phi_a(x)\,|\phi_a(x)\rangle\,\langle\phi_a(x)| = \mathbf{1}, \tag{95}$$

which can be obtained by the limit $\lim\limits_{N\to\infty}$ of the unit in the Hilbert space of N degree of freedom

$$\mathbf{1} = \int dq_1...dq_n|q_1...q_N><q_1...q_N| \text{ for all } t$$

by identifying the q_i's with the values ϕ_i on the space lattice and the continuum state $|\phi(x)\rangle = \lim\limits_{N\to\infty}|\phi_1...\phi_N\rangle$. One can also read this in a different way. (93) also shows the normalization

of the state functional, i.e. the wave function in the field representation, the exceedingly complicated object which one wants to get rid of:

$$\int \mathcal{D}\phi(x)\Psi^*[\phi(x)]\Psi[\phi(x)] = 1.$$

Thus one obtains for the spectral function:

$$K(T) = \text{Tr } e^{-\frac{i}{\hbar}\mathcal{H}} = \int \mathcal{D}\phi_a(x) K(\phi_a, T|_a, 0) = \int \mathcal{D}\phi_a(x) \int_{\phi_a,0}^{\phi_a,T} \mathcal{D}\phi(x,t) e^{\frac{i}{\hbar}S[\phi,T]}. \tag{96}$$

For the calculation of the functional integral we again use the stationary phase approximation. By the integration over the field space one also has to respect spatial boundary conditions for the fields.

Spatial boundary conditions. We shortly examine the influence of the spatial boundary conditions on (96) for theories in D=1+1 of the form

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - U(\phi) \tag{97}$$

where for all minima $U_{min} = 0$ is valid.

(i) unbroken symmetry:

In the case of a unique minimum of U the minimum should lie at $\phi \equiv 0$, which can always be reached by shifting the field. From (6) one can see, that for $\phi \equiv 0$ also the energy is zero. Thus we expect the quantum vacuum state $|vac\rangle_{[\phi]}$ at $\phi \equiv 0$, i.e. $|vac\rangle_{[\phi]} = |\phi(x,t) \equiv 0\rangle$. The boundary conditions for finite energy solutions are (7)

$$\phi(x \to \pm\infty, t) = 0 \tag{98}$$

and so the Fock space should only exists over functions satisfying (98), i.e. we have (Fock) states $|\psi\rangle_{[\phi]}$ which are located around functions $\phi(x \to \pm\infty) = 0$. Correspondingly in the functional integral one has to integrate only over fluctuations around a stationary point satisfying (98) for all t. Thus in the case of an unbroken symmetry one can only perform a perturbation theory around the vacuum, i.e. standard perturbation theory in our framework.

(ii) spontaneously broken symmetry:

In this case one has several minima $U(\phi_i) = 0$, $i = 1...M$ which gives rise to nontrivial topological sectors (see section 2.4). The boundary conditions for finite energy solutions classify the fields topological and are denoted by (7)

$$\phi(x \to \pm\infty, t) = \phi_{i^\pm}.$$

Since the different topological sectors are not connected the trace and thus the functional integral (96) has to be evaluated for each sector separately. The topological charge, which is not changed by "quantum fluctuations (see (29)), acts as a super-selection quantum number. One has to integrate over fields appropriate to the sector, i.e. over fluctuations around a stationary point of definite topological charge. The completeness relation (95) holds in the subspace according to the topological sector. This will be justified below, where we construct the Hilbert space of the nontrivial sector.

3.1.5 Quantum energy levels for static solitons

We now consider the SPA for theories of the form (97) which permits *one* static soliton solution $\phi_{cl} = \phi_{cl}(x)$ in a topological sector $\mathcal{S}$, i.e. we neglect the *zero-mode problem*[17]. We also restrict our considerations to a finite space region, which has to do with the regularization procedure (see below). Thus the action is given by

$$S[\phi] = \int_{\mathcal{B}} dtdx\mathcal{L}(\phi),$$

where $\mathcal{B} = LxT$ is the finite space-time region. The spectral function is given by the

Trace-cum-path integral

$$K(T) = \mathrm{Tr}_{\mathcal{S}}\, e^{-\frac{i}{\hbar}\mathcal{H}T} = \int_{\substack{\\ top.\ sector}} \mathcal{D}[\phi_a(x)] \int_{\phi_a,0}^{\phi_a,T} \mathcal{D}[\phi(x,t)] e^{\frac{i}{\hbar}S[\phi]}.$$

The first integration sums up all contributions of closed paths with start- and end point $\phi_a(x)$ in the considered topological sector. The second one adds all these contributions for all starting (end) points in the topological sector. The SPA will pick out the contribution of fields in the neighborhood of ϕ_{cl}. This set is characterized by the closed path condition $\phi(x,0) = \phi(x,T) = \phi_a(x)$ and that only second order deviations $O(\phi(x,t) - \phi_{cl})^2$ count.

Stationary phase approximation. We approximate the action (the phase), which is stationary for $\phi_{cl}(x)$, around this classical solution to evaluate the functional integral and the trace. Therefore we consider fields

$$\phi(x,t) = \phi_{cl}(x) + \eta(x,t).$$

Since $\phi_{cl}(x)$ is static, the closed-path condition implies for the fluctuations η:

$$\phi(x,0) = \phi(x,T) = \phi_a(x) \;\Rightarrow\; \eta(x,0) = \eta(x,T) = \eta_a(x).$$

The spatial BC will be determined below. They are essential ingredients of the regularization process. Expanding the action around the stationary point ϕ_{cl}, according the considered topological sector, one obtains

$$S[\phi,T] = S[\phi_{cl}+\eta] = S(\phi_{cl},T) - \frac{1}{2}\int_{LxT} dtdx\; \eta\left(\Box + U''(\phi_{cl})\right)\eta \tag{99}$$

$$+ \left[(\frac{1}{2}\partial\eta + \partial\phi_{cl})\eta\right]|_{\partial\mathcal{B}} + O(||\eta||^3). \tag{100}$$

The boundary terms are not vanishing, in contrast to the above sections, since the classical solution ϕ_{cl} are not exactly identical with the initial=final field configuration $\phi_a(x)$. Since ϕ_{cl} is static, the classical part of the action gives (using (6))

$$S(\phi_{cl},T) = -\int_0^T dt \int dx[\frac{1}{2}{\phi_{cl}}'^2 + U(\phi_{cl})] = -\int_0^T dt E[\phi_{cl}] = -E[\phi_{cl}]T = -M_{cl}T. \tag{101}$$

[17] If there are more, well separated classical solutions in a topological sector, the spectral function is the sum of each contribution. This case has to be distinguished from the existence of zero modes.

With the translation $\phi(x,t) \to \eta(x,t) = \phi(x,t) - \phi_{cl}$ of the integration variable we get for the trace

$$K(T)_{SPA} = e^{\frac{i}{\hbar}S(\phi_{cl})} \int \mathcal{D}[\eta_a(x)] \int_{\eta_a,0}^{\eta_a,T} \mathcal{D}[\eta(x,t)]\; e^{-\frac{i}{\hbar}\frac{1}{2}\int_{LxT} dtdx\; \eta(\Box + U'')\eta + \dots}. \tag{102}$$

The dots stands for the boundary term. The operator in the exponent,

$$\mathcal{O}(x,t) = \partial_t^2 - \partial_x^2 + U''|_{\phi_{cl}}(x) =: \partial_t^2 + \mathcal{SO}(x)$$

is separable since ϕ_{cl} depends only on x. Therefore we expand, analogous to the harmonic oscillator, the paths according to eigen-functions of the spatial part, but now with time dependent coefficients. So we have

$$\left(-\partial_x^2 + U''(\phi_{cl})\right)\xi_n(x) = \omega_n^2 \xi_n(x) \text{ with } \int_L dx \xi_m^* \xi_n = \delta_{m,n} \tag{103}$$

$$\eta(x,t) = \sum_n c_n(t)\xi_n(x) \text{ and } \eta_a(x) = \sum_n c_{a,n}\xi_n(x). \tag{104}$$

The operator in (103) is a Schrödinger operator, and thus the eigen-functions $\{\xi_n\}$ form a complete set. The spatial BC i.e., $\xi(-L/2), \xi(L/2)$, will be specified below. Also we leave the explicit form of the sums in (104) open, since this will also be part of the regularization procedure. So we get for the spatial part of the exponent in (102)

$$\int dx \eta(x,t)\mathcal{O}\eta(x,t) = \sum_{l,k} c_l^*(t) \left(\int dx \xi_l^*(x)(\partial_t^2 + \mathcal{SO})\xi_k(x) \right) c_k(t) \tag{105}$$

$$= \sum_l c_l(t)(\partial_t^2 + \omega_l^2)c_l(t). \tag{106}$$

[18]For the boundary term we assume that the spatial boundary conditions *do not* introduce any contributions, which will be justified in concrete calculations e.g. by the use of *topological boundary conditions* (see below). Nevertheless the time-like boundaries induce contributions, because the fluctuations are only closed paths and not periodic ones. With this assumption and the time-independence of ϕ_{cl} the boundary contribution in (100) is

$$\int_{\mathcal{B}} dtdx\; \partial_\mu[\partial^\mu(\phi_{cl} + \frac{1}{2}\eta)\eta] = \frac{1}{2}\int_{-L/2}^{L/2} dx \eta_a(x)[\dot\eta(t'',x) - \dot\eta(t',x)]$$

$$= \frac{1}{2}\sum_{l,k}\int_{-L/2}^{L/2} dx c_{a,k}^*[\dot c_l(t'') - \dot c_l(t')]\xi_k^*(x)\xi_l(x) = \frac{1}{2}\sum_l \int_{t'}^{t''} dt \partial_t(c_l \dot c_l).$$

Together with (106) this gives for the exponent in the path integral (102)

$$-\frac{1}{2}\sum_l \int_T dt c_l(t)(\partial_t^2 + \omega^2)c_l(t) + \frac{1}{2}\sum_l \int_T dt \partial_t(c_l\dot c_l) = \frac{1}{2}\sum_l \int_T dt \left(\dot c_l^2(t) - \omega_l^2 c_l^2(t)\right).$$

[18]In the second line we gave set $c_l^* = c_l$. From the eigen-functions in the appendix (7.1) one can see that the reality condition for the field η for the continuum modes is $c_l^* = c_{-l}$. But by a unitary transformation $c_l \to U_{lk}c_k$, which leaves the path integral invariant, one gets real oscillators c_l. In (106) and in the following, it is assumed that this transformation is already carried out, after the spatial integrations.

This is the sum of harmonic oscillators, each with the action as given by (81) and with the closed path BC $c_l(0) = c_l(T) = c_{a,l}$ which is the analogue of $q(0) = q(T) = q_0$ and therefore no further boundary contributions occur as shown for the harmonic oscillator above. The measure is therefore given by a product of harmonic oscillator measures

$$\mathcal{D}\eta(x,t) = \prod_l \mathcal{D}c_l(t) = \prod_l \left[B_N(T) \prod_{n=1}^{N} dc_{l,n} \right],$$

where for each oscillator the measure constant is the same, since it is independent of the oscillator frequency ω_l as discussed above. The occurrence of a zero mode, i.e. $\omega_l = 0$ must be treated separately, since in this case the action of this mode is no longer that of an harmonic oscillator but that of a free propagating particle in one dimension.

So we get for the trace (102)

$$K_{SPA}(T) = \mathrm{Tr}\ e^{-\frac{i}{\hbar}\mathcal{H}T}|_{SPA} = e^{-\frac{i}{\hbar}E[\phi_{cl}]T} \prod_l \int dc_{a_l} \int_{c_{a_l},0}^{c_{a_l},T} \mathcal{D}c_l(t)\ e^{\frac{i}{\hbar}\int_T dt(\frac{1}{2}\dot{c}_l^2 - \frac{1}{2}\omega_l^2 c_l^2)}, \qquad (107)$$

where the sum in the exponent is now written as product of exponentials.

Comments: (**I**) In the SPA the trace is a product of the classical part with an "infinite" set of harmonic oscillators. (**II**) the system was made discrete by the introduction of spatial BC which has to be specified and will be an essential part of the regularization procedure. They should be chosen in such a way that no spatial boundary contributions (100) to the action occur. This means that they should be "topological" (see below). (**III**) The derivation above is also valid for constant, i.e. trivial, classical solutions $\phi_{cl} = \phi_V = const.$ and therefore the trace for the vacuum sector gives analogous results. The only difference is the eigen-value problem (103), which in the case of a trivial solution is of course much simpler and gives different eigen-values ω^2. (**IV**) We have excluded the possibility of a zero mode $\omega^2 = 0$, which leads to subtle problems and must be investigated separately.

Energy levels. Assuming that no eigenvalue vanishes (no zero mode) we get with the result of the harmonic oscillator (88) for the spectral function (107) in the topological sector $\mathcal{S}$

$$\mathrm{Tr}_{\mathcal{S}}\ e^{-\frac{i}{\hbar}\mathcal{H}T}|_{SPA} = e^{-\frac{i}{\hbar}E[\phi_{cl}^{\mathcal{S}}]T} \prod_l \sum_{\nu_l=0}^{\infty} e^{-i\omega_l^{\mathcal{S}}(\nu_l+\frac{1}{2})T},$$

where l is the mode index and ν_l is the excitation index of the l'th mode. Thus a general state $|\{\nu_l\}\rangle$[19] has the energy-spectrum[20]

$$E_{\mathcal{S}}[\{\nu_l\}] = E[\phi_{cl}^{\mathcal{S}}] + \hbar \sum_l \omega_l^{\mathcal{S}}(\nu_l + \frac{1}{2}) + O(\hbar^2). \qquad (108)$$

This energy-spectrum formula is valid for trivial solutions ϕ_{cl}, like the vacuum, and nontrivial solutions, i.e. solitons. In both cases the lowest energy-level is given by the state where no

[19] That also with the nontrivial sector, i.e. for ϕ_{cl} a soliton solution, are quantum states associated will be considered below.

[20] The notion "energy-spectrum" should make clear that this is not the energy of the state $|\{\nu_l\}\rangle$, since there are some more ingredients for the energy like renormalization and zero-point energy (see below).

mode l is excited, i.e. $\nu_l = 0$ for all l. These are the ground states in the considered sectors $\mathcal{S}$ and given as

$$E_{\mathcal{S}} = E[\phi_{cl}^{\mathcal{S}}] + \frac{\hbar}{2}\sum_l \omega_l^{\mathcal{S}} + O(\hbar^2). \tag{109}$$

We have now all ingredients to calculate the energy correction for solitons except the renormalization contributions to the quantum-action. As we will see, the semi-classical approximation (=SPA) in the nontrivial sector (the soliton) is already a one-loop result, i.e. order $\hbar$, and thus one *has to* renormalize the theory to get control of UV-divergences. This will be considered in one of the following sections.

In a last comment we want to outline the quantum nature of the SPA-correction, beside the occurrence of $\hbar$. Since the quadratic part of the action (103) is exactly the stability equation (42), the functions $\phi_{cl} + c_n(t)\xi_n(x)$ are well behaved nearby classical solutions (see 41) if the eigen-value ω_n^2 is positive. The quantum nature of these fluctuations, beside that their action occurs as a phase, is that the oscillations are not classical oscillations $\sim e^{i\omega t}$ but treated as quantum-oscillators through the path integral in (107). This can also be seen from the full stability equation (40), where the classical nearby-solutions are those with "eigen-value" zero of this equation and for the diagonalization of the quadratic action in the path integral we are treating the eigen-modes of this equation with non-zero eigen-values.

3.1.6 The zero mode

We have shown above (section 2.6) that the occurrence of zero modes is connected with symmetries of the system and the classical solution. For our special models, the SG and ϕ^4 theory, such a zero mode occurs in the kink sector. It is the lowest eigen-value of the stability equation (7.1). These zero modes are connected with the translational symmetry of the kink solutions (35,20). For any position of the kinks x_0 the equations of motions are fulfilled. In both cases the zero modes are proportional to the spatial derivative of the kinks, i.e.

$$\eta_0(x) \sim \partial_x \phi_K(x),$$

and thus are a result of a small translation of the kink. In the path integral quantization this results in zero frequency $\omega_0 = 0$ in (107) and thus the associated degree of freedom c_0 is not a harmonic oscillator but rather like a free particle of unit mass. The reason for this is that the kink solution is only a local minimum of the action and the potential energy(density) $U(\phi)$, respectively. Thus a fluctuation in the symmetry direction does not change the energy and feels no restoring "force". Fluctuations transverse to this symmetry direction (in field space) feel the restoring force of the increasing potential, which is in first order that of a harmonic oscillator.

The trace integration of the zero mode gives a divergent result. The only closed path with initial = finial position c_{a_0} for a free propagating particle is the constant solution $c_{cl} = c_{a_0}$. The classical action $\int dt \frac{\dot{c}_0^2}{2}$ is zero for this solution. Thus the trace integration of the zero mode c_0 in (107) gives

$$\tilde{B}_N(T) \int dc_{a_0} \to \infty.$$

The factor $\tilde{B}_N(T)$ is the measure constant of a free unit mass particle. The breakdown of the SPA is no surprise, since for its validity we had required that two classical solutions are not too close to each other (77). But because of the translational invariance of the kink solutions we have a continuous family of solutions, parametrized by the kink position x_0. The free propagation of the zero mode degree of freedom corresponds to a *collective* motion of the kink and its *internal* quantum fluctuations. It is customary to treat the zero mode by the use of appropriate coordinates to describe the symmetry, called *collective coordinates*. This is analogous to for example atomic physics where the collective center of mass motion is separated from the internal motions described by relative coordinates. The idea is to find coordinates which describe the motion in the symmetry direction, i.e in the "valley" (surface) in field space which forms the relative minimum of the action, these are the collective coordinates. The method of SPA is only applicable to the residual coordinates, which describes the internal motion. The integration for the collective coordinates has to be carried out exactly. We demonstrate this for a simple integral:

$$I = \int d^n x \; e^{\vec{x}^2 - \lambda(\vec{x}^2)^2}. \tag{110}$$

The exponent is stationary for the "classical" solution

$$\vec{x}_{cl}(1 - 2\lambda \vec{x}_{cl}^2) = 0 \;\Rightarrow\; |\vec{x}| = \frac{1}{\sqrt{2\lambda}}.$$

This is an $n-1$ parameter family of solutions, each vector with the length $\frac{1}{\sqrt{2\lambda}}$. This corresponds to the $O(n)$ symmetry of the exponent in (110). If we single out one stationary (saddle) point and evaluate its contribution in a Gaussian approximation we would get $n-1$ zero eigenvalues and thus a catastrophic, divergent result. The solution to this problem is to use angular variables and integrate them exactly. Only for the radial variable one expands around the stationary point. Thus one obtains

$$I = \int d\Omega_{n-1} r^{n-1} dr \; e^{r^2 - \lambda r^4} = V_{S^{n-1}} \int dr r^{n-1} \; e^{r^2 - \lambda r^4}.$$

The radial integral can now be evaluated using Gaussian approximation. The angular integral is evaluated exactly and gives the volume of a $n-1$ dimensional sphere. Here the angular variables are the analogue of collective coordinates or cyclic coordinates, as they are called in classical mechanics.

The proper collective coordinate for the kinks is of course the position of the kink, which we will call $X(t)$. This means that we change from the coordinates $\{c_l(t); l = 0, \dots\}$ to coordinates $\{X(t), c_l(t); l = 1 \dots\}$ to get rid of the problematic zero mode $c_0(t)$. This is done by expanding the field according to

$$\phi(x,t) = \phi_K(x - X(t)) + \sum_{l=1} c_l(t) \xi_l(x - X(t)), \tag{111}$$

where ϕ_K and ξ are the same functions, i.e. the kink functions and the eigen-functions of the stability equation except for the zero mode, as before. That the new coordinate $X(t)$ consistently replaces c_0 and is thus independent of the other coordinates can be seen as follows: A small variation of $X(t)$ adds to $\phi(x,t)$ a term proportional to $\partial_x \phi_K(x - X(t))$, i.e.

$$\delta_X \phi(x,t) \approx \partial_x \phi_K(x - X(t)) \delta X.$$

But the derivative of the kink is proportional to the zero mode ξ_0, and thus

$$\int dx \partial_x \phi_K(x - X(t)) \xi_l(x - X(t)) = 0 \; \forall \; l.$$

Therefore a variation in the collective coordinate $X(t)$ is orthogonal to the other coordinate directions and the set $\{X(t), c_l(t); l = 1 \dots\}$ consists only of independent variables. Because of the translational invariance the new expansion (111) only changes the kinetic part of the action and the Lagrange function, respectively, i.e. the spatial integral of the Lagrangian:

$$\begin{aligned} L &= \int dx \mathcal{L} = \int dx \left[(\partial_t \phi)^2 - (\partial_x \phi)^2 - U(\phi) \right] \\ &= L_{Kin}(\dot{X}, \dot{c}_l) + L_{Pot}(c_l). \end{aligned}$$

Thus the Lagrange function is independent of the coordinate $X(t)$, it depends only on the velocity $\dot{X}(t)$. This is true for the full Lagrange function and action. Especially the quadratic part of L_{Pot} gives the same quadratic action as in (107) except for the zero mode $c_0(t)$. Since L depends only on the derivative $\dot{X}(t)$, the collective coordinate is a *cyclic* coordinate [14] and thus the proper coordinate to describe symmetries of the system. From the Euler-Lagrange e.o.m. it follows that the canonical momenta of cyclic coordinates are conserved, i.e. on a classical trajectory (on shell):

$$\frac{\partial L}{\partial \varphi_i} = 0 \Longleftrightarrow \varphi_i \; cyclic \Longleftrightarrow \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\varphi}_i} \right) = \frac{d}{dt} p_{\varphi,i} = 0.$$

Thus the canonical conjugated momenta of cyclic coordinates are the Noether charges of the associated symmetries. And as expected the canonical momentum of $X(t)$ is equal to the conserved total field momentum P [3], i.e. the Noether charge of the spatial translation symmetry:

$$\Pi_X = \frac{\partial L}{\partial \dot{X}} = P \text{ with } \frac{d}{dt} P = 0.$$

Since we are considering relativistic theories the classical energy of the kinks has the form

$$E(P) = \sqrt{M_{cl}^2 + P^2}. \tag{112}$$

Therefore the integration of the collective coordinate in the path integral gives only kinetic contributions which one can neglect in the considered order in (107) [3]. This means that the kink is effectively at rest. This is a reasonable approximation especially for the calculation of the quantum mass of the kink in this order . In higher orders the different modes in the spectral function are no longer independent. For (107) this means that the fluctuations for $l > 0$ interact with the zero mode fluctuation which is then no longer a free (zero) mode. But for the one-loop ($= SPA$) calculation of the kink masses we simply omit the integration over this mode. Thus, because of the translational zero mode, one has one mode less in the kink spectrum. This has a completely different origin than the discrete excited mode ξ_1 of the ϕ^4-kink (see appendix (7.1)) but an similar consequences in the *bosonic* case, as we will see.

The interplay between cyclic (collective) coordinates and the conserved associated momenta is to be expected to be more subtle in the case of constrained systems like fermions. And indeed we will see that this naive counting of the zero mode in the energy spectrum leads to wrong results in the case of fermions.

3.2 Standard perturbation theory and renormalization

We shortly survey the main points in the renormalization procedure. For simplicity we do this mostly for the ϕ^4 - model. One crucial point for a consistent renormalization is that one renormalizes the theory only once (at a given perturbation order) and uses this then fixed renormalized theory to calculate the desired quantities also in different topological sectors. It is customary to renormalize the theory by setting up renormalization conditions in standard perturbation theory, i.e. relations between scattering process contributions. This associates the parameters of the theory with certain measurable physical processes. So one can (must!) determine them by experiments. In renormalizable theories a finite number of such conditions also eliminates the divergences in the standard perturbation theory.

The standard perturbation theory is tailored for calculation of amplitudes in scattering processes, which are related to (vacuum) *n-point correlation (Green) functions* or their Fourier transformation, respectively, which are given as ($x = x^\mu$)

$$G_n(x_1, \ldots, x_n) = \langle\Omega| T\phi_H(x_1) \ldots \phi_H(x_n) |\Omega\rangle = \frac{\langle 0| T\phi_D(x_1) \ldots \phi_D(x_n)\, \mathcal{S} |0\rangle}{\langle 0| \mathcal{S} |0\rangle}. \tag{113}$$

The index H/D stands for the Heisenberg/ Dirac -picture and T is the time-ordering symbol. $\mathcal{S}$ is the S-matrix (operator) which is related to the time-evolution operator of the Schrödinger equation (50), ánd reads in the Dirac-picture as follows

$$\mathcal{S} := \lim_{\varepsilon \to 0_+} \mathcal{U}(-\infty, \infty) = \lim_{\varepsilon \to 0_+} T\, e^{-\frac{i}{\hbar}\int_{-\infty}^{\infty} dt \mathcal{H}^{\varepsilon}_D(t)}.$$

The limit $\varepsilon \to 0_+$ indicates the adiabatic "switch on" of the interaction, i.e. $\mathcal{H}^{\varepsilon}{}_D := \mathcal{H}_D e^{-\varepsilon|t|}$, and corresponds to the boundary conditions (preparation) for scattering processes (asymptotic free particles (fields)). For details see e.g. [26]. This is the reason why in standard perturbation theory soliton contributions are not seen. Of particular interest is the two-point function $G(x, y) = \langle\Omega| T\phi_H(x)\phi_H(y) |\Omega\rangle$:

3.2.1 Analytical structure of $G(x, y)$ and field strength renormalization

First we consider the spectrum of the Hamilton operator $\mathcal{H}$ and the momentum operator $\mathcal{P}_i$ (we consider a $D = 1 + 3$ space-time, so that $\mathcal{P}_i$ is a three-vector and $\mathcal{P}^\mu = \mathcal{P} = (\mathcal{H}, \mathcal{P}_i)^T$). Since they commute, i.e. $[\mathcal{H}, \mathcal{P}_i] = 0$, they have common eigen-states. The vacuum state $|\Omega\rangle$ is the eigen-state to the eigenvalue zero, i.e. $\mathcal{H}|\Omega\rangle = 0 = \mathcal{P}_i|\Omega\rangle$. Let $|\lambda_0\rangle$ be eigen-states of zero momentum, i.e.

$$\mathcal{H}|\lambda_0\rangle = m_\lambda |\lambda_0\rangle \tag{114}$$
$$\mathcal{P}_i|\lambda_0\rangle = 0, \tag{115}$$

then the boosted states $U_{boost}(\vec{p})|\lambda_0\rangle = |\lambda_{\vec{p}}\rangle$ are also eigen-states, but with momentum $\vec{p}$ and, because of relativistic invariance, energy $E(\vec{p}) = \sqrt{\vec{p}^2 + m_\lambda^2}$. Thus the eigen values m_λ are the energies in the rest-frame. In general the spectrum consists of the vacuum, the one-particle state ($m_\lambda = m$, particle mass), possible bound states ($m_\lambda = m_B$) and a continuum of multi-particle states (see fig.8). Thus the completeness relation, expressed in relativistically

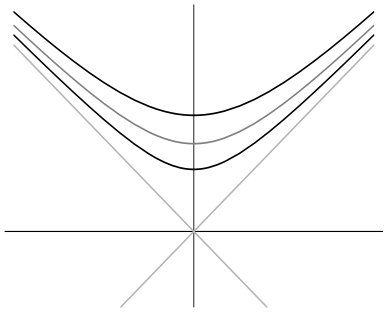


Figure 8: Th spectrum of $\mathcal{H}$ plotted against the spectrum of $\mathcal{P}$. At the origin sits the vacuum state $|\Omega\rangle$. On the ordinate lie the "masses" of the discrete one-particle- and bound- states, as well as a continuum of multi-particle- states (above the highest hyperboloid). For non-zero momentum they form hyperboloids according to the relativistic energy relation $E(p) = \sqrt{\vec{p}^2 + m_\lambda^2}$, which asymptotically approach to light cones. There my be also more bound states bellow the threshold of two free particles-creation.

normalized states, in the Hilbert space reads as [28]

$$\mathbf{1} = |\Omega\rangle \langle\Omega| + \sum_\lambda \int \frac{dp^3}{(2\pi)^3} \frac{1}{2E_\lambda(\vec{p})} |\lambda_{\vec{p}}\rangle \langle\lambda_{\vec{p}}| . \tag{116}$$

For the following we assume that $x^0 > y^0$. Inserting the unit (116) we thus get for the two-point function

$$G(x,y) = \langle\Omega| \phi_H(x)\phi_H(y) |\Omega\rangle \tag{117}$$

$$= \sum_\lambda \int \frac{dp^3}{(2\pi)^3} \frac{1}{2E_\lambda(\vec{p})} \langle\Omega| \phi_H(x) |\lambda_{\vec{p}}\rangle \langle\lambda_{\vec{p}}| \phi_H(y) |\Omega\rangle , \tag{118}$$

where we have dropped the uninteresting constant $\langle\Omega| \phi_H(x) |\Omega\rangle \langle\Omega| \phi_H(y) |\Omega\rangle$, which is usually zero [28]. The matrix elements in (118) can be written as follows

$$\begin{aligned} \langle\Omega| \phi_H(x) |\lambda_{\vec{p}}\rangle &= \langle\Omega| e^{i\mathcal{P}\cdot x} \phi_H(0) e^{-i\mathcal{P}\cdot x} |\lambda_{\vec{p}}\rangle \\ &= \langle\Omega| \phi_H(0) |\lambda_{\vec{p}}\rangle e^{-ip\cdot x}|_{p^0=E(\vec{p})} \\ &= \langle\Omega| \phi_H(0) |\lambda_0\rangle e^{-ip\cdot x}|_{p^0=E(\vec{p})}. \end{aligned}$$

In the first line we have written the field operator at space-time position x as a translation of the operator at the space-time origin. In the second line we have used the translational invariance, $\langle\Omega| e^{i\mathcal{P}\cdot x} = \langle\Omega| e^0$, of the vacuum and the Lorentz invariance of $\phi_H(0)$ and $\langle\Omega|$, i.e. $U_{boost}(\vec{p})\phi_H(0)U_{boost}^{-1}(\vec{p}) = \phi_H(0)$[21], so that $\langle\Omega| \phi_H(0)U_{boost}(\vec{p}) |\lambda_0\rangle = \langle\Omega| U_{boost}(\vec{p})\phi_H(0) |\lambda_0\rangle = \langle\Omega| \phi_H(0) |\lambda_0\rangle$.

Källén-Lehmann spectral representation. By introducing an p^0 -integration the momentum p becomes off-shell and the two point function can be written as $(x^0 > y^0)$

$$\langle\Omega| \phi_H(x)\phi_H(y) |\Omega\rangle = \sum_\lambda \int \frac{d^4p}{(2\pi)^4} \frac{i\, e^{-ip\cdot(x-y)}}{p^2 - m_\lambda^2 + i\epsilon} |\langle\Omega| \phi_H(0) |\lambda_0\rangle|^2. \tag{119}$$

[21]For fields with spin one has to respect the nontrivial internal transformations of the field, which are of the form $U\phi_\alpha U^{-1} = S_{\alpha\beta}\phi_\beta$. For spinor fields this gives the matrix structure $\not{p} + m$ in the propagator.

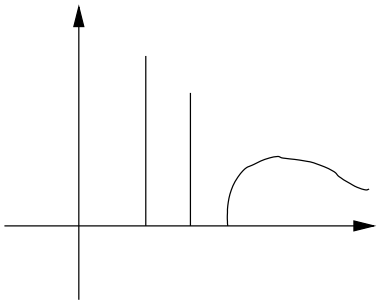


Figure 9: The spectral density for a typical interacting theory. The one-particle states contribute a delta function at m^2. Multi-particle states have a continuous spectrum, starting at $(2m)^2$. There may also discrete contributions of bound states.

Here the *Feynman propagator* $D_F(x-y, m_\lambda^2)$ appears but with m_λ instead of only the particle mass m. An analogous expression holds for $x^0 < y^0$ so that the full two point function is given by

$$G(x,y) = G(x,y) = \langle\Omega|\, T\phi_H(x)\phi_H(y)\,|\Omega\rangle = \int_0^\infty \frac{dM^2}{2\pi}\rho(M^2)D_F(x-y,M^2).$$

This is the Källén-Lehman spectral representation, where $\rho(M^2)$ is a positive spectral density function,

$$\rho(M^2) = \sum_\lambda 2\pi\delta(M^2 - m_\lambda^2)|\,\langle\Omega|\,\phi_H(0)\,|\lambda_0\rangle\,|^2,$$

For a typical interacting theory it is given by fig.9. Stable one-particle states $\left|\lambda_p^m\right\rangle$ contribute an isolated delta function to the spectral density:

$$\rho(M^2) = 2\pi\delta(M^2 - m^2)Z + \text{terms with } M^2 \geq m_B^2,$$

where $Z = |\,\langle\Omega|\,\phi_H(0)\,|\lambda_0^m\rangle\,|^2$ is the *field-strength renormalization* factor. The quantity m is the *exact* mass of a single particle, since it is the exact energy eigen-value of the full interacting theory, as can seen by (114). This quantity in general differs from the mass-parameter used in the Lagrangian (see below) and we refer to it the *physical mass* of the ϕ-boson.

A Fourier transformation of the two-point function (119) gives

$$\begin{aligned}\int dx^4 e^{ip\cdot x}\,\langle\Omega|\, T\phi_H(x)\phi_H(0)\,|\Omega\rangle &= \int_0^\infty \frac{dM^2}{2\pi}\rho(M^2)\frac{i}{p^2 - M^2 + i\epsilon} \\ &= \frac{iZ}{p^2 - m^2 + i\epsilon} + \int_{\sim m_B^2}^\infty \frac{dM^2}{2\pi}\rho(M^2)\frac{i}{p^2 - M^2 + i\epsilon} \quad .\end{aligned}$$

The analytical structure of the Fourier transformed two-point function is as follows: The first term gives a simple pole at $p^2 = m^2$ with the residue Z, while the second term contributes a branch cut beginning at $p^2 = (2m)^2$ and additional poles for possible bound states below the cut. Thus contributions of from one-particle and multi-particle intermediate states can be distinguished by the strength of their analytic singularities. This analysis relies only on general principles of relativity and quantum mechanics , it does not depend on the nature of interaction or on perturbation theory, except that we have used scalar fields due to notational

simplicity. The only input of standard perturbation theory is that we have considered the vacuum-correlation function and thus soliton contributions do not occur, since the Hilbert space built around the soliton is not connected with the "vacuum" Hilbert space. There exists no operator which connects states between these two sectors (see below). This analysis generalizes to higher order n-point functions. The analytical structure shows that the (Fourier transformed) n-point functions are the multi-particle (field theoretical) analogue of the kernel resp. the Green function of section 3.1.1.

In the case of free fields, or zeroth order perturbation theory, the Fourier transformed two point function writes as

$$\int dx^4 e^{ip\cdot x} \langle 0| T\phi_D(x)\phi_D(0) |0\rangle = \frac{i}{p^2 - m^2_{free} + i\epsilon}.$$

For $x^0 > 0$ this can be interpreted as the amplitude that a particle created at the space-time position $y = 0$ propagates to x. It is similar to the full two-point function except two differences: There are no multi-particle contributions since free fields can only create single particle states. The field strength renormalization constant $Z = |\langle \Omega| \phi_H(0) |\lambda^m_0\rangle|^2$, i.e. the probability for $\phi_H(0)$ to create an exact one particle state, is in the free case equal to one, i.e. $\langle p| \phi_D(0) |\rangle = 1$. Note that by a *renormalization* of the field strength this probability can be normalized to one also for the interacting theory:

$$\phi_H \to \phi^{ren}_H := \frac{1}{\sqrt{Z}}\phi_H \;\Rightarrow |\langle \Omega| \phi^{ren}_H(0) |\lambda^m_0\rangle|^2 = \frac{1}{Z}|\langle \Omega| \phi_H(0) |\lambda^m_0\rangle|^2 = \frac{Z}{Z} = 1.$$

With this renormalization also the residue of the single-particle pole in the two-point function (propagator) is normalized to one:

$$\int dx^4 e^{ip\cdot x} \langle \Omega| T\phi^{ren}_H(x)\phi^{ren}_H(0) |\Omega\rangle = \frac{1}{Z}\int dx^4 e^{ip\cdot x} \langle \Omega| T\phi_H(x)\phi_H(0) |\Omega\rangle = \frac{i}{p^2 - m^2 + i\epsilon} + \dots$$

3.2.2 The systematics of renormalization

Primarily the renormalization has nothing to do with the occurrence of divergences (in perturbation theory), but to express the theory (action) in terms of measurable quantities, i..e. in terms of parameters which are related to certain (reference) experiments. In he following we assume that the theory is regularized in some way, so that all considered quantities are well defined. This we will indicate with an index Λ. We indicate the unrenormalized (not adjusted to the reference-experiments) parameters with an index 0 to distinguish them from the renormalized ones (parameters which are related to the reference experiments). The selected reference experiments and their relations to the renormalized parameters define the values of these parameters in the measurement and are called *renormalization conditions*. For simplicity we consider the ϕ^4 theory without spontaneous symmetry breaking, The fields can be viewed as ordinary functions, since we will use the path integral for further calculations:

$$\mathcal{L}_\Lambda = \frac{1}{2}(\partial\phi_0)^2 - \frac{1}{2}m^2_0\phi^2_0 - \frac{\lambda_0}{4!}\phi^4_0. \tag{120}$$

This Lagrangian and the associated action has all symmetries which are not destroyed by the regularization. We have seen that we can renormalize the field-strength. Therefore we set

$$\phi = \frac{1}{\sqrt{Z_1}}\phi_0 \;\Leftrightarrow\; \phi_0 = \sqrt{Z_1}\phi, \tag{121}$$

where the constant Z_1 need not be equal to Z above, although this would be a natural choice. Its actual value will be defined by a renormalization condition. One can also use the freedom in fixing Z_1 to make calculations simpler instead of getting a simple relation to the reference experiment. With this first renormalization the Lagrangian writes as

$$\mathcal{L}_\Lambda = \frac{1}{2}Z_1(\partial\phi) - \frac{1}{2}m_0^2 Z_1\phi^2 - \frac{\lambda_0}{4!}Z_1^2\phi^4.$$

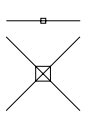
We have just inserted (121) in the Lagrangian, thus it is still the same theory. Next we split the mass and coupling parameter into a renormalized (measured) part and a part which will be fixed by the renormalization conditions in concrete calculations, i.e. we divide them into a part which will be measured and a part which will be calculated. We also write Z_1 as such a sum, i.e we define

$$m_0^2 Z_1 := m^2 + \delta m^2 \tag{122}$$
$$\lambda_0 Z_1^2 := \lambda + \delta\lambda \tag{123}$$
$$Z_1 := 1 + \delta Z_1. \tag{124}$$

Inserting this into the Lagrangian we get

$$\mathcal{L}_\Lambda = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 + \frac{1}{2}\delta Z_1(\partial\phi)^2 - \frac{1}{2}\delta m^2\phi^2 - \frac{\delta\lambda}{4!}\phi^4. \tag{125}$$

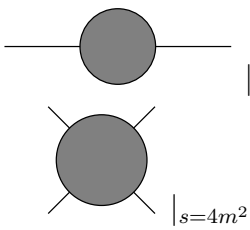
Of course this Lagrangian has the same symmetries as (120), i.e. those symmetries which despite regularization are still present. The Lagrangian consists of the "classical" part, expressed in terms of the renormalized parameters, and *counter terms* (the δ-terms) which are treated as interaction terms in the perturbation theory. Thus classical properties, as for example the classical kink masses (22) in the model considered above, are not affected. Also the fundamental ingredient of the standard perturbation theory, the Feynman propagator (see below), is not affected, since it is derived from the quadratic part of the classical action. In this view the counter terms are the quantum contributions to the full quantum action (125) relative to the classical action which is expressed in renormalized parameters. The additional interaction terms, the counter terms, lead to additional *Feynman graphs* in the perturbation theory:



$= i(p^2\delta Z_1 - \delta m^2)$

$= -i\delta\lambda$

Until now the splittings of the parameters (122) - (124) are purely formal. To give them a physical meaning we have to set up *renormalization conditions* which relate the renormalized parameters to certain scattering processes so that they can be determined by experiments. As an example we choose:



$|_{p^2\to m^2} = \frac{i}{p^2-m^2} + (\text{terms regular at } p^2 = m^2) \quad \Rightarrow \delta m^2,\ \delta Z$

$|_{s=4m^2,\, t=u=0} = -i\lambda \quad \Rightarrow \delta\lambda$

These relations for the "scattering amplitudes" displayed by the graphs above determine the explicit form of the counter terms. Especially the first renormalization condition is very "natural", since it renormalizes the residue (the probability of a single particle creation of the vacuum) to one and sets the renormalized mass parameter m to the pole of the single particle contribution of the two point function. As we have seen above, this is the *exact* physical particle mass, i.e. the eigen-value to the one-particle eigen-state of the full Hamiltonian (114). But there is no need to choose these "physical" renormalization conditions. One can also choose less simple relations between the two-point function and the renormalized parameters, which are more comfortable for other calculations for example. The second renormalization condition has no "natural" or obviously best definition. The renormalization condition is set up for the scattering amplitude at zero momentum, which means in Mandelstam variables $s = 4m^2$ and $t = u = 0$. The values of the variables p, s, t, u at which the renormalization conditions are defined are called *renormalization point.* Different renormalization points leads to different *renormalization schemes.* They are of course all equivalent, since a different renormalization point only changes the division between the renormalized parameters and the counter term constants in (122) - (124). One can always put a finite piece of e.g. m^2 into δm^2. Thus a different renormalization point (different renormalization condition) only results in

$$m^2 + \delta m^2 \to \tilde{m}^2 + \delta\tilde{m}^2 = (m^2 + \Delta) + (\delta m^2 - \Delta).$$

This of course gives the same results but expressed in terms of $\tilde{m}^2$.

Renormalizable theories. We were cheating as we said that the renormalization procedure has nothing to do with the occurrence of divergences, since this procedure was developed to control divergences in the perturbation theory. The classical part of the Lagrangian (125) produces divergent contributions (if the regularization is removed) in the perturbation theory. Thus also the counter terms must be divergent (without regularization) to compensate the divergent contributions by the renormalization conditions. A theory is called renormalizable if all divergences in the perturbation theory can be compensated by a *finite* number of renormalization conditions, thus the results are finite even if the regularization is removed at the end of the calculation. In a perturbative expansion the counter terms must be determined order by order, this means in each order one has to solve the renormalization conditions, given above.

Symmetries. If all divergences can be compensated by counter terms obtained by renormalization of the parameters, as above, the quantum corrections do not violate any of the symmetries which are left in the regularized theory $\mathcal{L}_\Lambda$. The question is weather the symmetries which are broken by the regularization are reestablished when the regularization is removed after calculations. If a symmetry can not be reestablished one calls this an *anomaly.* In a perturbative expansion in $\hbar$ (loop-expansion) each symmetry of the theory, which does not involve $\hbar$, is fulfilled in each order, i.e. if one respects all contribution up to the considered order. If not all divergences are canceled by counter terms of the form of the Lagrangian one needs additional counter terms, which must be added "by hand", e.g. a term $C\phi^3$. The additional parameters must be determined by experiment and the symmetry of the theory is probably broken.

3.2.3 Renormalization of ϕ^4- and SG- model

Generating functional. The n-point vacuum correlation functions (113) can be written as a path integral as follows [28]

$$G_n(x_1,\ldots,x_n) = \lim_{T\to\infty(1-i\epsilon)} \frac{\int \mathcal{D}\phi\ \phi(x_1)\ldots\phi(x_n)\ e^{\frac{i}{\hbar}\int_{-T}^{T} dx^4\mathcal{L}}}{\int \mathcal{D}\phi\ e^{\frac{i}{\hbar}\int_{-T}^{T} dx^4\mathcal{L}}}. \tag{126}$$

The denominator is the analogue of $\langle 0|\,\mathcal{S}\,|0\rangle$ in (113) , which connects the Dirac-vacuum $|0\rangle$ with the Heisenberg-vacuum $|\Omega\rangle$. The path integrals have to be evaluated in the vacuum sector, this means that

$$\lim_{T\to\infty(1-i\epsilon)} \int \mathcal{D}\phi = \int_{\phi_\Omega,-T}^{\phi_\Omega,T} \mathcal{D}\phi = \lim_{T\to\infty(1-i\epsilon)} \int_{\phi_V,-T}^{\phi_V,T} \mathcal{D}\phi. \tag{127}$$

The n-point function (126) can be written as functional derivatives of a generating functional

$$G_n(x_1,\ldots x_n) = \frac{1}{Z[0]}\left(\frac{\hbar}{i}\right)^n \frac{\delta^n Z[j]}{\delta j(x_n)\ldots\delta j(x_1)}\ |_{j=0}, \tag{128}$$

where the generating functional is given by

$$Z[j] = \int \mathcal{D}\phi\ e^{\frac{i}{\hbar}\int d^4x\left(\mathcal{L}+j\phi+i\epsilon\frac{\phi^2}{2}\right)}, \tag{129}$$

and must be calculated perturbatively for nontrivial theories. The $i\epsilon$-term is the analog of the small imaginary part of the time in (126) to ensure vacuum boundary conditions. It acts as a damping factor and gives the pole-description for the Feynman propagator. In the following we will suppress this term in our notation.

ϕ^4 **-model**. The full quantum action of the ϕ^4 model (15) is the Lagrangian expressed in unrenormalized parameters

$$\mathcal{L}_\Lambda = \frac{1}{2}(\partial\phi_0) - \frac{\lambda_0}{4}(\phi_0^2 - \frac{\mu_0^2}{\lambda_0})^2. \tag{130}$$

Since in two dimensions only the mass receives a divergent contribution (see below, "educated guessing") it is enough to choose a minimal renormalization scheme:

$$\delta Z = 0 \Rightarrow \phi_0 = \phi, \qquad \delta\lambda = 0 \Rightarrow \lambda_0 = \lambda, \qquad \mu_0^2 = \mu^2 + \delta\mu^2.$$

Since we have to evaluate the path integral (126) in the vacuum sector, and to fulfill the asymptotic boundary conditions (127), we expand the Lagrangian (action) around one of the classical vacua (19). We choose $\phi_V = \frac{\mu}{\sqrt{\lambda}}$ so that $\phi = \frac{\mu}{\sqrt{\lambda}} + \eta$. Thus by this perturbation theory one will never "see" soliton contributions. Also inserting $\mu_0^2 = \mu^2 + \delta\mu^2$ we get[22]

$$\mathcal{L}_\Lambda = \frac{1}{2}[(\partial\eta)^2 - 2\mu^2\eta^2] - \sqrt{\lambda}\mu\eta^3 - \frac{\lambda}{4}\eta^4 - \frac{1}{2}\delta\mu^2\left(\eta^2 + \frac{2\mu}{\sqrt{\lambda}}\eta\right) + O(\hbar^2). \tag{131}$$

[22]Since we are considering an unbounded space-time, no surface terms contribute to the action in this expansion.

We have put the constant $\frac{(\delta\mu^2)^2}{4\lambda}$ into the higher order terms indicated by $O(\hbar^2)$, since we will determine $\delta\mu^2$ only in one loop ($\hbar$) order. The physical boson mass at tree level is $m = \sqrt{2}\mu$, as one can read off of the quadratic part of the Lagrangian, and has the correct sign. The vacuum boundary condition (pole prescription) in (129) is respected implicitly by a small imaginary part of the squared mass, i.e. $m^2 \equiv m^2 - i\epsilon$. With this Lagrangian we get for the generating functional (129), but with the source coupled to the "physical" field η

$$Z[j] = \int \mathcal{D}\eta \, e^{\frac{i}{\hbar}\int dx^4\left(\frac{1}{2}[(\partial\eta)^2 - m^2\eta^2] + j\eta + \mathcal{L}_I\right)} \tag{132}$$

$$= \int \mathcal{D}\eta \left[\sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{i}{\hbar}\right)^k \left(\int dx^4 \mathcal{L}_I\right)^k\right] e^{\frac{i}{\hbar}\int dx^4\left(\frac{1}{2}[(\partial\eta)^2 - m^2\eta^2] + j\eta\right)}. \tag{133}$$

This is the perturbative expansion, where $\mathcal{L}_I$ is the "quantum" interaction Lagrangian

$$\mathcal{L}_I = -\sqrt{\lambda}\mu\eta^3 - \frac{\lambda}{4}\eta^4 - \frac{1}{2}\delta\mu^2\left(\eta^2 + \frac{2\mu}{\sqrt{\lambda}}\eta\right) + O(\hbar^2).$$

The perturbative expansion can also be written as functional derivatives of the *free generating functional*

$$Z[j] = e^{\frac{i}{\hbar}\int dx^4 \mathcal{L}_I\left(\frac{\hbar}{i}\frac{\delta}{\delta j}\right)} Z_0[j] \text{ with } Z_0[j] = \int_\Lambda \mathcal{D}\eta \, e^{\frac{i}{\hbar}\int dx^4\left(\frac{1}{2}[(\partial\eta)^2 - m^2\eta^2] + j\eta\right)}, \tag{134}$$

where the index Λ indicates that the free generating functional has to be evaluated in a regularized way. The regularization takes place in the set of considered fluctuations η, i.e. the "path integration domain" PID. We consider two possibilities which are very similar in the trivial sector.

Energy-momentum cutoff. We restrict the PID to a regularized one, which is characterized by the cutoff Λ or the projector, respectively,

$$\hat{\delta}(x) = \int \frac{dk^2}{(2\pi)^2}\theta(\Lambda - |k_1|)\, e^{ik\cdot x},$$

which is the (regularized) unit in the regularized domain $PID_{Reg} = \{\eta \mid \hat{\delta}\cdot\eta = \eta\}$. Also the sources $j(x)$ must be functions in this domain, i.e. $\int dy^2 \hat{\delta}(x-y)j(y) = j(x)$. The regularized set of fluctuations PID_{Reg} are functions which have a compact support in the spatial Fourier transformed variable, i.e.

$$\eta \in PID_{Reg} \;\Rightarrow\; \eta(x) = \int \frac{dk^2}{(2\pi)^2}\theta(\Lambda - |k_1|)\, \tilde{\eta}(k)\, e^{ik\cdot x}.$$

The free generating functional (134) can be evaluated in several ways. One is to expand the fluctuations η around the configuration η_0, i.e. $\eta \to \eta_0 + \eta$, where η_0 fulfills:

$$(\Box + m^2)\eta_0(x) = j(x) \tag{135}$$

$$\Rightarrow\; \eta_0(x) = -\int dy^2 \Delta_F^{reg}(x-y)j(y) \text{ with } (\Box + m^2)\Delta_F^{reg}(x) = \hat{\delta}(x). \tag{136}$$

The differential equation for the free field in (135) is solved by the method of Green functions. The Green function $\Delta_F^{reg}(x)$ in (136) is called *Feynman propagator* and is characterized

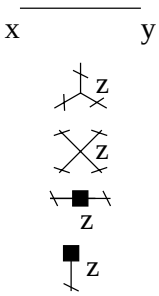
by the vacuum boundary conditions which is encoded in the pole prescription. Its Fourier representation is given by

$$\Delta_F^{reg}(x) = \int \frac{dk^2}{(2\pi)^2}\theta(\Lambda - |k_1|)\frac{e^{-ik\cdot x}}{k^2 - m^2 + i\epsilon}.$$

We have written the pole prescription explicitly. Inserting (136) into the free generating functional (134) one obtains

$$Z_0[j] = N\, e^{-\frac{i}{\hbar}\frac{1}{2}\int dx^2 dy^2\; j(x)\Delta_F^{reg}(x-y)j(y)}, \tag{137}$$

where the constant N is a number which will be canceled by the denominator in (128). From (134) and with the form of the interaction Lagrangian (131) a general n-point function is a composition of Feynman graphs which are given by:



$i\hbar\Delta(x-y)$	propagator
$\frac{i}{\hbar}\sqrt{\lambda}\mu\int dz^2$	3-vertex
$-\frac{i}{\hbar}\lambda\int dz^2$	4-vertex
$-\frac{i}{\hbar}\delta\mu^2\int dz^2$	seagull-counter-term
$\frac{i}{\hbar}\frac{\mu}{\sqrt{\lambda}}\delta\mu^2\int dz^2$	tadpole-counter-term

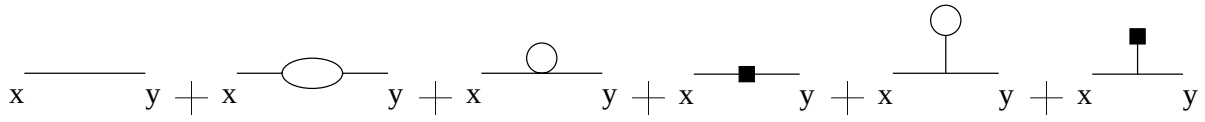
Because of the spontaneous symmetry breaking also a three-vertex interaction occurs. The integrations for the vertices lead to momentum conservation in momentum space. The *disconnected* graphs are canceled by the denominator $Z[0]$ in (128). This gives the usual Feynman rules (see for example [29]). The order in $\hbar$ of an amputated graph is as follows (external lines are replaced by wave functions in the S-matrix) [27]: Because of the expansion of the exponential function in (133), each vertex V contributes a factor $\hbar^{-1}$. From each internal line I, $\sim \hbar^2\frac{\delta^2}{\delta j^2}\frac{1}{\hbar}\int j\Delta j$, comes a factor $\hbar$. Therefore $\frac{i}{\hbar}W[j] = O(\hbar^{I-V})$, where $Z[j] = e^{\frac{i}{\hbar}W[j]}$. The functional $W[j]$ is the generating functional for connected Green functions. Therefore the contribution of an connected amputated graph to $W[j]$ is proportional to

$$\hbar^{I-V+1} = \hbar^{\ell},$$

where ℓ is the loop number of this graph.

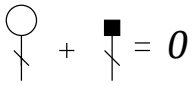
Renormalization conditions. With the Feynman rules the two-point function has the following graphical representation:

$$G(x,y) = \tag{138}$$



$$+O(\hbar^2)$$

In two dimensions only the seagull (third graph) and the tadpole-pole (fifth) graphs are divergent if the regularization is removed. We fix $\delta\mu^2$ by the requirement that the tadpole vanishes exactly. This means



$$\Rightarrow\ i\hbar 3\sqrt{\lambda}\mu\Delta(0)+\frac{\mu}{\sqrt{\lambda}}\delta\mu^2\stackrel{!}{=}0 \tag{139}$$

Thus the mass-counter term is given by

$$\delta\mu^2=-i\hbar 3\lambda\Delta(0)=-i\hbar 3\lambda\int\frac{dk^2}{(2\pi)^2}\frac{\theta(\Lambda-|k_1|)}{k^2-m^2+i\epsilon}=\frac{3\lambda\hbar}{2\pi}\int_0^\Lambda\frac{dk}{\sqrt{k^2+m^2}}, \tag{140}$$

where in the last step we have carried out the k_0-integration and set $k^1=k$. This is the one loop counter term and from (140) one can see that it is of order $O(\hbar)$ and depends on the regularization cutoff Λ. By this renormalization condition also the seagull-self-energy graph is canceled.The sum of the seagull- and the mass-counter- graph (graph three and four in 138) reads as

$$(i\hbar)^2 3\lambda\Delta(0)+i\hbar\delta\mu^2=(i\hbar)^2 3\lambda\Delta(0)-(i\hbar)^2 3\lambda\Delta(0)=0.$$

This renormalization scheme is the most simple one from the technical point of view, but one must not forget that the second graph in (138), a three vertices self energy diagram, gives a finite contribution to the pole of the one loop propagator. Thus the renormalized parameter m, defined by the renormalization condition (139) is *not* the pole of the propagator. For the physical one-loop mass, i.e. the pole of the propagator, one must take this finite contribution into account. Summing up the series in (138) as usual [29] the loop of the three vertex self energy diagram contributes to the pole. Thus the physical one loop mass is given by [37]

$$m_P^2=m^2+9\lambda i\hbar\int\frac{dk^2}{(2\pi)^2}\frac{m^2}{(k^2-m^2+i\epsilon)((k-p)^2-m^2+i\epsilon)}\,|_{p^2\to m^2}=m^2-\frac{\sqrt{3}}{2}\hbar\lambda.$$

We also could have renormalized the coupling, e.g. through low-energy limit of scattering amplitudes, i.e. an analogous renormalization condition as in (3.2.2), to get simpler relations between the renormalized parameters and physical measured quantities. But our renormalization condition (139) is an equivalent scheme and for concrete calculations is the most comfortable one.

The renormalization condition (139) gives the explicit expression for the counter term parameter $\delta\mu^2$ and defines the renormalized parameter μ resp. $m=\sqrt{2}\mu$. Its concrete value must be determined by experiment of course. But all quantities expressed through this parameter get their explicit meaning by the renormalization condition. With the explicit expression for $\delta\mu^2$ and the definition of μ through the renormalization condition also the one-loop renormalized Lagrangian (130) has an explicit meaning ($\phi_0=\phi,\ \lambda_0=\lambda,\ \mu_0^2=\mu^2+\delta\mu^2$:

$$\mathcal{L}_\Lambda=\mathcal{L}(\mu)+\delta\mathcal{L}_\Lambda,$$

where $\mathcal{L}(\mu)$ is the classical Lagrangian expressed through the renormalized parameter, and $\delta\mathcal{L}_\Lambda$is the (one-loop) counter term Lagrangian, the quantum corrections, which in our scheme is given by

$$\delta\mathcal{L}_\Lambda=\frac{\delta\mu^2}{2}\left(\phi^2-\frac{\mu^2}{\lambda}\right)+O(\hbar^2). \tag{141}$$

The classical action can be thought of as renormalized at zero loop level, with the renormalization condition $\delta\mu^2 = 0$. Therefore all classical quantities are expressed through the *finite,* regularization independent, parameter μ^2. This parameter must of course also be determined by experiment, for example by classical scattering amplitudes of solitons. In general this is a hypothetic issue, since the classical meaning of a quantum field theory is not always evident (see for example fermions).

Mode number cutoff. The regularized evaluation of the fundamental ingredient of the perturbation theory, the free generating functional $Z_0[j]$ and thus the Feynman propagator Δ_F^{reg}, can also be carried out in a discrete manner. For this one compactifies the spatial dimension to a circle of (large) perimeter L or considers a compact interval of length L and introduces appropriate boundary conditions (periodic or antiperiodic are proper ones, see below). The path integration domain PID_{Reg} is then defined by a finite Fourier expansion according to the discrete Fourier modes. In the vacuum sector there is no big difference between an energy momentum cutoff (EMC) and a mode number cutoff (MNC). Only the finite spatial momentum integrations are replaced by finite sums, i.e. one has the correspondence

$$\int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} f(k) \;\longleftrightarrow\; \sum_{-N}^{N} f(k_n) \text{ with } Lk_n = (2n + A)\pi.$$

The value of $A = 0, 1$ depends on the boundary conditions. The two expressions coincide up to order $O(\frac{1}{L})$ and the two cutoffs EMC and MNC are related as

$$\Lambda = \frac{(2N + A)\pi}{L}.$$

These relations between EMC-integrals and MNC-sums are not so simple in the soliton sector as we shall see. But the counter term parameter $\delta\mu^2$ (140), defined in the vacuum sector is simply

$$\delta\mu^2 = 3\lambda\hbar \frac{1}{2}\frac{1}{L} \sum_{-(N-A)}^{N} \frac{1}{\sqrt{k_n^2 + m^2}} = \frac{3\lambda\hbar}{2\pi} \int_0^{\Lambda} \frac{dk}{\sqrt{k^2 + m^2}} + O(\frac{1}{L}, \frac{1}{\Lambda}).$$

Therefore we will always write the counter term in the integral representation, independent of the used regularization scheme.

Renormalization of the sine-Gordon model. Since we were so explicit above we can treat the SG-model relatively quickly. The full quantum Lagrangian, i.e. expressed in unrenormalized parameters, is given by

$$\mathcal{L}_\Lambda = \frac{1}{2}(\partial\phi)^2 + \frac{\mu_0^2}{\gamma}\left[\cos(\sqrt{\gamma}\phi) - 1\right],$$

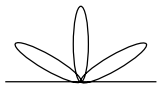
where we have used that only the mass gets divergent contributions (if the regularization is removed), as we shall see. Thus we have the following minimal renormalization scheme:

$$\delta Z = 0, \qquad \delta\gamma = 0, \qquad , \mu_0^2 = \mu^2 + \delta\mu^2.$$

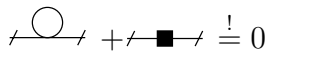
Expanding the Lagrangian around the vacuum $\phi = 0$ and inserting $\mu_0^2 = \mu^2 + \delta\mu^2$ one obtains

$$\mathcal{L}_\Lambda = \frac{1}{2}\left[(\partial\eta)^2 - \mu^2\eta^2\right] + \mu^2\frac{\gamma}{4!}\eta^4 - \mu^2\frac{\gamma^2}{6!}\eta^6 \cdots - \frac{\delta\mu^2}{2}\eta^2 + \ldots. \tag{142}$$

Only the seagull loops, which are for example of the form



are divergent [37], and thus our minimal renormalization scheme is enough to get rid off of the divergences. The tree-level boson mass is μ as one can see from the quadratic part of (142). At one-loop the renormalization condition, that the seagull loops vanishes reads as



$$\Rightarrow \quad \delta\mu^2 = \hbar\frac{\mu^2\gamma}{2}\int_0^\Lambda \frac{dk}{2\pi}\frac{1}{\sqrt{k^2+\mu^2}}.$$

Since at one-loop order the one-loop seagull graph is the only contribution, the mass parameter μ, defined by the above renormalization condition, coincides at one loop with the pole of the propagator, i.e. the physical mass of the boson. The one loop counter term Lagrangian is given by

$$\delta\mathcal{L}_\Lambda = \frac{\delta\mu^2}{\gamma}\left[\cos(\sqrt{\gamma}\phi) - 1\right] + O(\hbar^2). \tag{143}$$

The renormalization condition that all seagull loops vanish gives the complete counter term as follows [5]

$$\delta\mathcal{L}_\Lambda = -\frac{\mu^2}{\gamma}\left(e^{\delta\mu^2/\mu^2} - 1\right)\left[\cos(\sqrt{\gamma}\phi) - 1\right].$$

3.2.4 Quantum action for solitons

With the counter term contributions $\delta\mathcal{L}_\Lambda$ and the explicit expressions the counter term parameters $\delta\mu$ we can write down the renormalized (quantum) action in the soliton sector. The classical kink masses (22,36) and all other quantities are expressed through the renormalized parameters μ, which are defined by the renormalization conditions from above. It is important to use the action renormalized in the vacuum sector, also to calculate the desired quantities in the soliton sector, since this ensures that if one compares quantities of the two sectors one talks about the same things, i.e. one uses the same renormalized parameters, defined by the same renormalization conditions. The one-loop mass counter terms for the SG- and ϕ^4- model, as calculated above are given by

$$\delta\mu^2_{SG} = \hbar\frac{\gamma m^2}{4\pi}\int_0^\Lambda \frac{dk}{\sqrt{k^2+m^2}} \tag{144}$$

$$\delta\mu^2_{\phi} = \hbar\frac{3\lambda}{2\pi}\int_0^\Lambda \frac{dk}{\sqrt{k^2+m^2}}, \tag{145}$$

where m is the tree-level boson mass, and related to the renormalized mass parameter μ by ($l = 1/2$ for SG/ϕ^4)[23]

$$m = \sqrt{l}\mu.$$

[23]For the SG-model it is even the one-loop physical (pole) boson mass.

The one-loop counter term Lagrangians are given by (143), (141)

$$\delta\mathcal{L}_{SG}(\delta\mu) = \frac{\delta\mu^2}{\gamma}\left[\cos(\sqrt{\gamma}\phi) - 1\right] + O(\hbar^2) \tag{146}$$

$$\delta\mathcal{L}_{\phi}(\delta\mu) = \frac{\delta\mu^2}{2}\left(\phi^2 - \frac{m^2}{2\lambda}\right) + O(\hbar^2), \tag{147}$$

these counter term Lagrangians lead to additional contributions to the action also in the kink sector. To calculate these contributions we expand (146),(147) around the kink solutions (20,35) of the two models ($\sigma = 1/-1$ for kink/antikink),

$$\phi^4: \quad \phi_{K_\sigma} = \sigma\frac{m}{\sqrt{2\lambda}}\tanh\left[\frac{m(x-x_0)}{2}\right] \tag{148}$$

$$SG: \quad \phi_{K_\sigma} = \frac{4}{\sqrt{\gamma}}\arctan[e^{\sigma m(x-x_0)}], \tag{149}$$

and keep terms up to order $O(\hbar)$. For the SG kink we have re-introduced the original coordinate and field relatively to (35). For nontrivial classical solutions the classical action is non-zero, in contrast to the vacuum solutions, and therefore already gives contributions of order $O(\hbar^0)$. Thus the quadratic fluctuations (the semi-classical amplitude) are already of order $O(\hbar)$. So the nontrivial contribution of the classical action $S[\phi_K]$ shifts the order by one, relatively to the vacuum sector, in the expansion of quantum fluctuations. Thus the one-loop counter term contribution in the kink sector is given by

$$\delta\mathcal{L}(\phi_K + \eta; \delta\mu) = \delta\mathcal{L}(\phi_K; \delta\mu) + O(\hbar^2).$$

The interaction of the fluctuations with the counter term $\sim \delta\mu = O(\hbar)$ give higher order contributions. Thus the one-loop quantum corrections to the action in the kink sector are given by

$$\delta S_{SG}(\delta\mu) = \frac{\delta\mu^2}{\gamma}\int_{LxT} dtdx\ \left[\cos(\sqrt{\gamma}\phi_{K_\sigma}) - 1\right] + O(\hbar^2) = -T\frac{4}{m\gamma}\delta\mu^2 + O(\hbar^2)$$

$$\delta S_{\phi}(\delta\mu) = \frac{\delta\mu^2}{2}\int_{LxT} dtdx\ \left(\phi^2_{K_\sigma} - \frac{m^2}{2\lambda}\right) + O(\hbar^2) = -T\frac{m}{\lambda}\delta\mu^2 + O(\hbar^2).$$

For SG we have used that $\delta\mathcal{L}_{SG} = -\frac{\delta\mu^2}{\mu^2}U(\phi)$ and the mass formula for static solutions. In the SPA calculation of the energy spectrum this gives additional contributions to the mass of static solutions. In addition to the classical mass of static solutions (101), such as kinks, the counter terms give, besides the quantum fluctuations, contributions to the quantum mass. For SG and ϕ^4 kinks this gives

$$S_{cl}(\phi_K) + \delta S(\phi_K) = -T\left(M_{cl} + \delta M(\delta\mu)\right) \text{ with :} \tag{150}$$

$$SG: \quad \delta M_{SG} = \frac{4}{m\gamma}\delta\mu^2 \tag{151}$$

$$\phi^4: \quad \delta M_{\phi} = \frac{m}{\lambda}\delta\mu^2. \tag{152}$$

Now we have all ingredients to calculate the one-loop quantum corrections to the classical masses of static solitons like the SG and ϕ^4 kinks.

4 Quantum masses of static solitons

We have now all ingredients to calculate the quantum corrections to the classical kink masses (22,36) in a semi-classical approximation, i.e. at one loop order. The one loop corrections to the masses, especially for the supersymmetric extension of the here considered models (see below), are of particular interest, since they are connected with the possible occurrence of an anomaly in the supersymmetry algebra. The main question is whether the Bogomolnyi bound stays saturated by $N = 1$ supersymmetric solitons including quantum corrections resp. whether there exists an anomaly in the central charge of the SUSY algebra ([37],[36],[34]). This can be decided already at one-loop level.

As one can see from (109) the ground state energies include for both, the vacuum and the kink sector, divergent sums over the mode energies $\omega_l^{\mathcal{S}}$. This corresponds to the ambiguity (freedom) in the choice of the energy zero point in ordinary (no gravity) quantum field theory (only energy differences are measurable). The absolute energy-zero point must be fixed for the vacuum ground-state. The natural choice is to normalize vacuum ground-state to zero energy. Thus we have to subtract the zero-point energy of the vacuum (no state is excited) from the energy calculated for any state. As long as one considers only the (undistorted) vacuum sector this procedure is trivial and respected by the normal ordered Hamiltonian, i.e.

$$: \mathcal{H} := \mathcal{H} - \langle 0| \mathcal{H} |0\rangle \quad \langle 0| : \mathcal{H} : |0\rangle = 0.$$

Thus the vacuum ground-state $|0\rangle$ has zero energy. The same reference-point for the energy must be used for all other states, also for the soliton[24], i.e.

$$\langle sol| : \mathcal{H} : |sol\rangle = \langle sol| \mathcal{H} |sol\rangle - \langle 0| \mathcal{H} |0\rangle .$$

To evaluate this difference in the presence of a nontrivial background such as a kink is a highly nontrivial issue and object of controversial discussions for years (see [37] and references therein). Thus the one-loop kink ground-state energy is given by

$$E_K - E_V,$$

where the ground-state energies E_K, E_V are given by (109). In the discrete version, as given in (109), one has to evaluate the difference of two infinite, divergent sums. Thus they (the theory) must be regularized in a consistent manner. This will be the subject of the following sections. The calculation of the difference of these ground-state energies is very similar to the Casimir effect but much more involved, since the kink background is of course much more complicated than conducting plates, which set up certain boundary conditions.

4.1 Renormalized spectral function

To evaluate the spectral function (102) in one-loop order one has to do a semi-classical expansion of the one-loop *renormalized* action[25] $S = \int_{T\text{x}L} dtdx \left(\mathcal{L}(m) + \delta\mathcal{L}(\delta\mu)\right)$ around the classical

[24]That with the soliton also a quantum state is associated ill be discussed below.

[25]Here we introduced the interval L for the spatial integration. The range of L depends on the regularization scheme and will be finite or equal $\mathbb{R}$

solution, i.e. $\phi(x,t) = \phi_{cl} + \eta(x,t)$ up to order $O(\hbar)$. The classical quantities are expressed in terms of renormalized parameters. For static solutions ϕ_{cl} one obtains

$$S[\phi_K + \eta(x,t)] = -\left(M(\phi_K) + \delta M(\delta\mu)\right) T \tag{153}$$

$$-\frac{1}{2}\int\limits_{T\mathrm{x}L} dtdx\ \eta(x,t)\left(\Box + U''(\phi_{cl})\right)\eta(x,t) + O(\hbar^2). \tag{154}$$

The first two terms in (154) are the classical energy (i.e. the classical mass for static solutions) and the counter-term contribution and given by the renormalized Lagrangian with the counter terms (141,143). In the vacuum sector ($\phi_{cl} = \phi_V = const$) both terms vanish. In the kink sector one obtains the classical kink mass M_{cl} and the counter-term contribution to the quantum correction of the mass (151,152)

$$\delta M_{SG} = \frac{4\delta m^2}{\gamma m} = 2\hbar \int_0^\Lambda \frac{dz}{2\pi}\frac{1}{\sqrt{z^2+1}} \tag{155}$$

$$\delta M_{\phi^4} = \frac{m\delta m^2}{\lambda} = 3\hbar \int_0^\Lambda \frac{dz}{2\pi}\frac{1}{\sqrt{z^2+1}}. \tag{156}$$

In the integrals (156) we have transformed to the variable $z = \frac{k}{m}$ so that the now dimensionless cutoff Λ is large (or small) relative to the scale m. The linear term in (154) is absent since ϕ_{cl} is a classical solution. In dimensionless spatial coordinates $z = \frac{m}{l}x$ the spatial part of the operator in (154) for the vacuum and the kink sector, respectively, is given as ($l = 1/2$ for SG/ϕ^4) [26]

$$Vacuum: \qquad O_{V_l}(z) = (-\partial_z^2 + l^2), \tag{157}$$

$$Kink: \qquad O_{K_l}(z) = \left(-\partial_z^2 + l^2 - \frac{l(l+1)}{\cosh^2 z}\right). \tag{158}$$

Therefore in the kink sector one has exactly the stability operator (see appendix). To evaluate the spectral function one has to diagonalize these two operators and one is left with the following expression

$$\mathrm{Tr}_S[\mathrm{e}^{-\frac{i}{\hbar}HT}] = e^{-\frac{i}{\hbar}(M_{cl}+\delta M)T}\int\limits_{\{PID_{reg}\}}\mathcal{D}\eta_a \int_{\eta_a,t'}^{\eta_a,t''}\mathcal{D}\eta\ e^{-\frac{i}{\hbar}\frac{1}{2}\int\limits_{T\mathrm{x}\tilde{L}} dtdz\frac{m}{l}\eta[\partial_t^2 + O_S(z)]\eta} + O(\hbar^2) \tag{159}$$

The exponent of the first factor in (159) is only non-zero in the kink sector. The only difference to (102) is the counter term contribution. With $\{PID_{Reg}\}$ (for "path integration domain") we indicate that the set of considered paths and therefore the spectral function depends on the regularization which will be used. This will be discussed in detail in the next sections. In the quadratic part of the action the field- degrees of freedom are the fluctuations η, the classical solution ϕ_{cl} is a fixed background which is nontrivial in the kink sector.

4.2 Mode regularization (MNC) for bosonic kinks

Mode regularization proceeds by making the system discrete by a finite volume L and thus countable (so one is not involved with functional analytical subtleties as in the continuum)

[26] In principle there are also surface terms because of the finite time interval T (see above). But at the end they are transformed back to total derivative to get the harmonic oscillator action. This is a trivial step and we suppress it here.

and introduces a mode number cutoff (MNC) to regularize otherwise divergent expressions. This is very analogous to a lattice regularization and, at least for bosons, there exists a one-to-one correspondence even though both schemes are not related by a normal coordinate transformation in the path integral [22]. There a two main points in a MNC-scheme that become crucial in the presence of a nontrivial background: (I) the boundary conditions should not induce effects that do not vanish in the limit $L \to \infty$. (II) The number of discrete states and the correct evaluation of the sums over mode energies. The relation between the "cutoffs" in different topological sectors is given by the requirement that in both sectors an equal number of modes is taken into account. Thus the two Hilbert spaces PID^{Vac}_{Reg} and PID^{Kink}_{reg} have the same dimension and therefore this two spaces are isomorphic. Thus the correct counting of the discrete states is essential for mode regularization.

(I) Boundary Conditions

In those cases where boundary conditions are essential in the regularization process, we adopt a new principle, which is closely related to but less restrictive than the "topological boundary conditions" of [36]. To ensure that the BC do not introduce a force which contributes to the energy we compactify the spatial direction to a circle of perimeter L ($\tilde{L} = \frac{m}{l}L$ in our coordinates). The fields must therefore fulfill a matching condition which leads to certain BC, depending on the topology of the line bundle one chooses. According to (159) the fields that have to fulfill this matching condition are the fluctuations η. First of all the resulting BC must be a linear relation so that the considered set of paths in (159) form a linear space. We require now that the BC must be chosen in a way that the transport of the quadratic Lagrangian $\mathcal{L}^{(2)}(\eta)$ around the compactified dimension leaves $\mathcal{L}^{(2)}(\eta)$ invariant. This means

$$z \to z + \tilde{L}: \quad \mathcal{L}^{(2)}(\eta) \to \mathcal{L}^{(2)}(\eta) \Rightarrow \delta\mathcal{L}^{(2)}(\eta) = 0$$

Otherwise the action would get an additional contribution $\sim \int_{\tilde{L}} dz \delta\mathcal{L}^{(2)}$ by the spatial integration. Thus the topology of the line bundle on which η lives must be chosen in a way so that a surrounding of the compact dimension ($z \to z + \tilde{L}$) induces a linear symmetry transformation of $\delta\mathcal{L}^{(2)}$. In both topological sectors the influence of the classical solution (158)is symmetric, thus on the circle one has $O_S(z + \tilde{L}) = O_S(z)$. Therefore in both sectors one can use the following line bundles:

$$z \to z + \tilde{L} \;\Rightarrow\; \eta(z + \tilde{L}) = A\eta(z) \quad A = \pm 1$$

The values of $A = \pm 1$ corresponds to periodic P and anti periodic AP BC. This is the $\mathbb{Z}_2$ symmetry which is despite spontaneous symmetry breaking conserved in the quadratic part of the Lagrangian. The BC can be chosen independently in both sectors, all combinations $(Vac|kink) = (P, AP|P, AP)$ are allowed since our symmetry principle ensures that no contributions due to the BC occur. There is no need to use common BC in both sectors (in contrary to [31],[32]). This is the physical principle for mode regularization. It is rather simple and not restricted to two dimensional theories. In the case of fermions it will become more exciting.[27]

[27]Also homogeneous BC are allowed is by the symmetry principle although they are not topological. For the sake of simplicity we do not consider them here although in principle possible.

4.2.1 Regularized kink mass

To carry out the path integral in (159) one has to diagonalize the quadratic action. In a mode regularization scheme this is done by a finite expansion of the fluctuations according to eigen-functions of the operator $O_S(z)$. For this one has to solve the eigen-value problem

$$\left(-\partial_z^2 + O_S(z)\right)\xi_n = \omega_n^2 \xi_n \tag{160}$$

with proper boundary conditions. A *finite* expansion of the fluctuations according to the eigen-functions of (160)

$$\eta(x,t) = \sum_{-M_-}^{M_+} c_n(t)\xi_n(x) \tag{161}$$

leads to a finite, countable set of harmonic oscillators with eigen-frequencies ω_n (see above). The *mode number cutoffs* M_-, M_+will be determined for each special case below. Therefore the path integral in (159) reduces to a finite product of harmonic oscillators of frequency ω_n:

$$\prod_{M_-}^{M_+} \int dc_{a_n} \int_{c_{a_n},t'}^{c_{a_n},t''} B_n(T)\mathcal{D}c_n(t)\; e^{\frac{i}{\hbar}\int_T dt(\frac{1}{2}\dot{c}_n^2 - \frac{1}{2}\omega_n^2 c_n^2)} \tag{162}$$

The measure $B_n(T)$ is the same for each oscillator n, independent of the sector: the measure of a harmonic oscillator. Thus for an equal number of modes in the vacuum- and kink sector one has the same measure in both sectors. There is a subtlety connected with possible zero modes. For the zero mode integration one has to use collective coordinates. In the purely bosonic case this is a fairly trivial thing and connected with breaking of the translation invariance by a given kink position. As showed above, in the considered order one can omit the integration of the zero mode.[28]

The residual integration is easily performed. Putting all things of (159) together one obtains for the difference between the kink ground state energy and the vacuum ground state energy the following one loop kink mass

$$M_K = E_K - E_V = M_K^{cl} + \frac{\hbar}{2}\sum_{bound}\omega_b^K + \frac{\hbar}{2}\left(\sum_{-M_-}^{M_+}\omega_n^K - \sum_{-N_-}^{N_+}\omega_n^V\right) + \delta M(\delta m) + O(\hbar^2). \tag{163}$$

The mode energies $\omega_n^{K,V}$are given by the eigen values of (160) for the kink (K) resp. vacuum (V) sector.

[28]In the supersymmetric case this issue is much more involved, since fermionic zero modes have the same origin and should be treated by a common collective coordinates (see below).

4.2.2 Vacuum contribution

For the vacuum the calculations and results are rather simple and in both cases given by ($A_V = 1/-1$ for P/AP BC):

$$\text{BC} - \text{quantization} \qquad Lmk_n = (2n+\pi)n \tag{164}$$

$$\text{groundstate energy}: \qquad E_V = \frac{\hbar m}{2} \sum_{-(N-A_V)}^{N} \sqrt{\left(\frac{(2n+A_V)\pi}{Lm}\right)^2 + 1} \tag{165}$$

$$\text{mode number}: \qquad \#_V = 2N + 1 + A_V \tag{166}$$

$$\text{energy cutoff}: \qquad \Lambda_A = k_N^A = \frac{(2N+A_V)\pi}{Lm} := \Lambda + \frac{A_V \pi}{Lm}. \tag{167}$$

In the vacuum the conversion of the sum (167) into an integral is straightforward. Nevertheless we use the Euler-MacLaurin formula which will be the appropriate technique in the kink sector. So one obtains

$$E_V = \frac{\hbar m}{2} \left\{ (1 - A_V) + 2 \int_{\frac{(2-A_V)\pi}{Lm}}^{\Lambda + \frac{A_V \pi}{Lm}} \frac{dz}{2\pi} Lm\sqrt{z^2+1} + \left(1 + \sqrt{\Lambda^2+1}\right) \right\} + O(\frac{1}{L}, \frac{1}{\Lambda}). \tag{168}$$

In the last term within braces, the "surface term", we have already carried out the limit $L \to \infty$.

4.2.3 Kink sector

In the kink sector all calculations become much more involved. First we outline the general principles. In the following subsections we carry out the calculation for the considered models. A very sensitive point is the evaluation of the (potentially) infinite sum. The appropriate and mathematically exact tool to do this is the Euler-MacLaurin formula, which is given in the appendix (318).

In both cases the continuum states (314),(315) are asymptotically of the form

$$\xi(q, z \to \pm\infty) = Z^{\pm}(q)\, e^{iqz} = |Z^{\pm}| e^{i\left[qz + \varphi^{\pm}(q)\right]} \tag{169}$$

where $Z^{\pm}(q)$ are complex valued functions of the momentum q. For SG and ϕ^4, respectively, they have the explicit form

$$Z^{\pm}_{sG}(q) = \pm 1 - iq \tag{170}$$

$$Z^{\pm}_{\phi^4}(q) = (2 - q^2) \mp i3q. \tag{171}$$

The absolute values $|Z^{\pm}|$ are not interesting (can be absorbed in the normalization) but the argument functions $\arg[Z^{\pm}(q)] =: \varphi^{\pm}(q)$ will become very important. Their explicit forms depend in a crucial way on the position of the branch cut chosen for the argument function.

From the asymptotic form (169) one can see that going once around the (large) space-circle one picks up a total phase

$$q\tilde{L} + [\varphi^+(q) - \varphi^-(q)] =: q\tilde{L} + \delta(q). \tag{172}$$

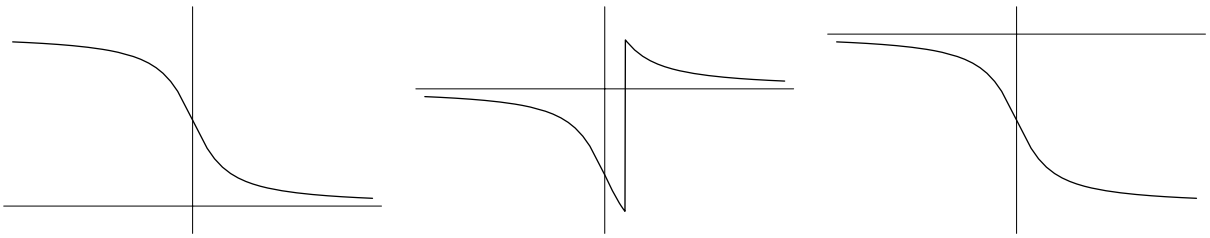


Figure 10: The Sine-Gordon scattering phase $\delta(q)$ for different branch cuts. (a) $cut = Im_+$, $\arg(z) \in (-\frac{3\pi}{2}, \frac{\pi}{2}]$. (b) $cut \neq Im_\pm$, $\arg(z) \in (\alpha, \alpha + 2\pi]$. (c) $cut = Im_-$, $\arg(z) \in (-\frac{\pi}{2}, \frac{3\pi}{2}]$.

where $\tilde{L}$ is the perimeter in z-coordinates from above, i.e. $\tilde{L} = \frac{m}{l}L$. The analytical structure of the scattering phase $\delta(q) := \varphi^+(q) - \varphi^-(q)$ and its asymptotic values depend on the position of the branch cut. By setting up boundary conditions the momenta get quantized as follows

$$q_n \tilde{L} + \delta(q_n) = (2n + A_K)\pi \tag{173}$$

This is a transcendental equation for the allowed q_n's and we will solve them by iteration . The constant A_K again determines the kind of boundary conditions ($A_K = 1| - 1$ for $P|AP$ BC). To apply the Euler-MacLaurin formula (318) to evaluate the ground state energy

$$E_K = \frac{\hbar m}{2} \sum_{-M_-}^{M_+} \sqrt{(\frac{q_n}{l})^2 + 1}$$

we have to know the addend as an explicit function of n. Therefore we have to resolve (173) to get q_n as an explicit function of n, at least up to sufficient orders in L and the mode number cutoff $M_\pm$. This is done by an iteration, where the second step gives

$$q_n = \frac{(2n + A_K)\pi}{\tilde{L}} - \frac{1}{\tilde{L}}\delta(\frac{(2n + A_K)\pi}{\tilde{L}}) + O(\frac{1}{\tilde{L}^2}). \tag{174}$$

That this is a reasonable approximation is guaranteed by the Banach fixed point theorem. The scattering phase δ is bounded for all values of n (its maximum range is 4π as the difference of two angles (172)) and therefore the iteration (174) is a contraction if $\tilde{L} > \delta_{max}$. This is of course true since we are interested in the limit $\tilde{L} \to \infty$. Next we do the explicit calculations for the SG and ϕ^4 model.

4.2.4 sine-Gordon-kink

From the asymptotic states (314) one obtains for the scattering phase

$$\delta(q) = \varphi^+(q) - \varphi^-(q) = -2 \arctan q + Cut$$

where Cut stands for branch cut position dependent constants. The scattering phase varies in range of 2π. Its branch cut dependent shape is given in fig.10. With the position of the branch cut also the discontinuity moves. At the discontinuity the phase jumps by 2π and this is also the vertical distance between two neighboring straight lines $(2n + A_K)\pi - \tilde{L}q_n$ which must intersect the graph δ for q_n being a solution of (173). Therefore in the case of a discontinuous phase always one mode has no solution and thus does not exist.[29]

[29]Note that at the discontinuity the phase takes only one value which corresponds to the semi-open intervals for the range of angles

From the continuous version of the scattering phase one can obtain by Levinson's theorem the number of discrete states as

$$\frac{\delta(-\infty) - \delta(\infty)}{2\pi} = \#_{discrete} = 1.$$

This is exactly the translational zero mode of the SG kink, which counts as a full mode.[30] Therefore the continuous spectrum in the kink sector is shifted down by one mode relative to the vacuum sector. For a discontinuous phase it is a low lying mode (in the sense that $M_\pm \to \infty$) which has no solution, whereas the phase goes asymptotically to zero ($\delta(\pm\infty) = 0$). For a continuous phase one has to omit one of the high modes, but now the phase takes asymptotically the finite value 2π. This interplay between the asymptotic values of the scattering phase and the omitted modes explains that even though a mode at the threshold of the continuum states becomes a "bound" state it is possible to subtract a high or a low mode of the continuum. These are the general rules which apply in all considered cases, also for fermions.

We calculate the kink mass for the discontinuous phase with the branch cut at $\mathbb{R}_+$, so that the phase jumps at $q = 0$ and is symmetric.[31] For a more general branch cut position the calculation is quite analogous but one has to write a little bit more. The case of a continuous phase will be considered in the ϕ^4 model. We leave the combination of boundary conditions of the two sectors arbitrary. For the sum over "continuous" kink mode energies we make a symmetric *ansatz*

$$\sum_{-(N+A_K),\neq 0}^{N} \omega_n^K.$$

Therefore the "continuous" mode numbers are given as follows

$$\text{vacuum}: \qquad \#_V = 2N + 1 + A_V \tag{175}$$

$$\text{kink}: \qquad \#_K = 2N + A_K. \tag{176}$$

Now independent of the BC one must have $\#_V - \#_K = 1$ due to the discrete zero mode in the kink sector. For equal BC ($A_V = A_K$) one can see that this is already fulfilled. In the case of different BC one has to add ($A_K = 0$ and $A_V = 1$) or subtract ($A_K = 1$ and $A_V = 0$) in addition one of the high modes. So different BC compensate the effect of the discontinuity of the scattering phase. Thus for the kink modes we have

$$\sum_{-(N+A_K),\neq 0}^{N} \sqrt{q_n^2 + 1} + (A_V - A_K) m \sqrt{\Lambda^2 + 1}.$$

For the additional high mode we have already taken the limit $L \to \infty$, but a detailed calculation shows that the result is independent of the sequence of the limits $L \to \infty$ and $\Lambda \to \infty$. Inserting the iterative solution (174) and using the Euler-MacLaurin formula we get with the

[30] This not as obvious as it seems. This will become clear in the case of fermions.

[31] We choose the semi-open interval for angles so that the phase takes the negative value at $q = 0$. Therefore in both cases P/AP BC the mode $n = 0$ has no solution.

variable transformation $z = \frac{(2n+A_K)\pi}{Lm}$

$$\sum_{-(N+A_K),\neq 0}^{N} \omega_n^K = mA_K + 2\sum_{1}^{N}\omega_n^K + (A_V - A_K)m\sqrt{\Lambda^2+1}$$
$$= mA_K + 2m \int_{\frac{(2+A_K)\pi}{Lm}}^{\Lambda+\frac{A_K\pi}{Lm}} \frac{dz}{2\pi} Lm \sqrt{\left(z - \frac{1}{Lm}\delta(z)\right)^2 + 1}$$
$$+m\left(1+\sqrt{\Lambda^2+1}\right) + (A_V - A_K)m\sqrt{\Lambda^2+1}$$

In the surface term we have already carried out the limit $L \to \infty$. As a next step we expand the root $\sqrt{z^2 - \frac{2z}{Lm}\delta(z) + O(\frac{1}{L^2}) + 1}$, where $\frac{2z}{Lm}\delta(z)$ is, for all z, a small quantity if L is large enough, since δ goes to zero sufficiently fast. But for branch cut positions where the phase takes finite asymptotic values one has to take care at this point. Let us consider such an example for the ϕ^4 model. Putting all together we get with (168) and (156) for the kink mass

$$M = \quad M_{cl} + \frac{\hbar m}{2}(A_K - (1 - A_V)) + \hbar m \left(\int_{\frac{(2+A_K)\pi}{Lm}}^{\Lambda+\frac{A_K\pi}{Lm}} - \int_{\frac{(2-A_V)\pi}{Lm}}^{\Lambda+\frac{A_V\pi}{Lm}} \right) \frac{dz}{2\pi} Lm\sqrt{z^2+1}$$
$$+\frac{\hbar m}{2}(A_V - A_K)\sqrt{\Lambda^2+1} - \hbar m \left(\int_0^\Lambda \frac{dz}{2\pi} \frac{z\delta(z)}{\sqrt{z^2+1}} + \int_0^\Lambda \frac{dz}{\pi} \frac{1}{\sqrt{z^2+1}} \right) + O(\frac{1}{L}) + O(\hbar^2)$$

In the second line we have already carried out $L \to \infty$ since the integrand is independent of L. The surface terms of the Euler-MacLaurin formula have canceled each other. This is always the case, so we do not write them down in the further calculations. Doing the integrals and taking the limit $L \to \infty$ one obtains

$$M_{cl} + \frac{\hbar m}{2}\left(A_K + A_V - 1 - (A_K + A_V) + (A_K - A_V)\sqrt{\Lambda^2+1} + (A_V - A_K)\sqrt{\Lambda^2+1}\right)$$
$$-\hbar m \frac{1}{2\pi}(2-\pi).$$

So the divergences cancel each other nicely and we finally obtain for the kink mass

$$M_K = M_{cl} - \frac{\hbar m}{\pi} + O(\hbar^2)$$

We have been so explicit to show that even in the case of different BC the correct mode counting gives the correct and finite result. The calculations for the continuous scattering phase are quite analogous and give exactly the same result.

4.2.5 ϕ^4-kink

In the ϕ^4 model everything is straightforward, but more involved since ϕ^4 - kink is one "degree" higher than the SG - kink[32]. From the asymptotic states (315) one obtains for the scattering

[32] SG- and ϕ^4- kink are two special cases of a class of kinks whose zero mode is of the form $\sim \frac{1}{\cosh^l z}$[41]. In this sense SG/ϕ^4 is of degree $l = 1/2$.

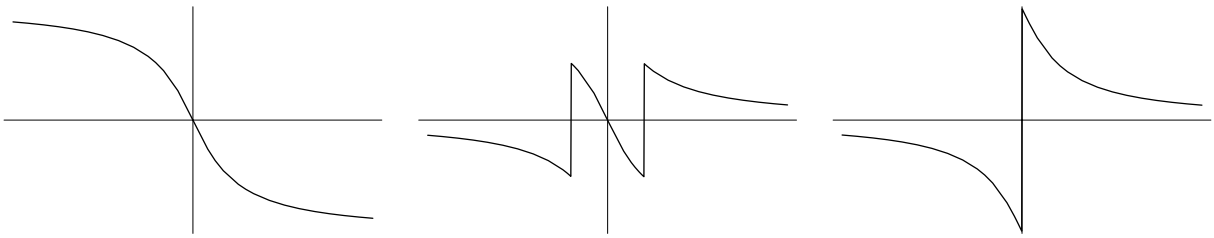


Figure 11: The ϕ^4 scattering phase for different branch cuts: (a) $cut = \mathbb{R}_-$, $\arg(z) \in [-\pi, \pi)$. (b) $cut \neq \mathbb{R}_- \vee \mathbb{R}_+$, $\arg(z) \in [\alpha, \alpha + 2\pi)$. (c) $cut = \mathbb{R}_+$, $\arg(z) \in [0, 2\pi)$. With the position of the branch cut the discontinuities are moving.

phase

$$\delta(q) = -2 \arctan\left(\frac{3q}{2 - q^2}\right) + Cut \tag{177}$$

Cut again stands for branch cut dependent contributions. The scattering phase (177) takes its values always in the semi open interval $[-2\pi, 2\pi)$ (if the upper or lower interval bound is the open one depends on the convention). For different branch cut positions the phase has the form as shown in fig.11 For discontinuous phases two modes do not have a solution and therefore do not occur in the sum over mode energies. The information on the discrete modes is again encoded in its asymptotic values. Levinson's theorem gives for the number of discrete states

$$\frac{\delta(-\infty) - \delta(\infty)}{2\pi} = \#_{discrete} = 2$$

These are the zero mode and the excited bound state of the ϕ^4 kink (315). Therefore the continuous spectrum of the kink shifts down by two modes relative to the vacuum.

We now calculate the kink mass using the continuous phase to show how to deal with the non-zero asymptotic values of the phase. We again do not fix the BC combination of the two sectors to show that there is no need to use the same BC in the two sectors. The above consideration (176) showed that the subtlety in mode counting only depends on the use of different or equal BC. Thus, to reduce the notational costs, we choose periodic BC for the vacuum and let the BC in the kink sector unspecified. For the sum over "continuous" kink mode energies we again make a symmetric *ansatz*

$$\sum_{-(N-1+A_K)}^{N-1} \omega_n^K$$

Therefore the "continuum" mode numbers are given as follows

$$\begin{aligned} \text{vacuum}: \quad & \#_V = 2N + 1 \\ \text{kink}: \quad & \#_K = 2N - 1 + A_K \end{aligned}$$

Now independently of the BC in the kink sector one must have two continuum modes less than in the vacuum sector, i.e. $\#_V - \#_K = 2$. In the case of equal BC (here periodic, i.e. $A_K = 0$) this is already fulfilled. For different BC, i.e antiperiodic BC in the kink sector ($A_K = 1$) one

has to subtract in addition one of the high modes. Therefore with the use of (315) we get for the kink sum

$$\sum \omega_n^K = -(1-A_K)m + m\left(2\sum_0^{N-1}\sqrt{(\frac{q_n}{2})^2+1} - A_K\sqrt{(\frac{q_N}{2})^2+1}\right) \tag{178}$$

The (dimensionless) momenta q_n are given by the iterative solution (174) and therefore the root in (178) can be written as ($\tilde{L} = \frac{m}{2}L$)

$$\begin{aligned}&\sqrt{\left(\frac{(2n+A_K)\pi}{Lm} - \frac{1}{Lm}\delta(2\frac{(2n+A_K)\pi}{Lm})\right)^2+1}\\ &= \sqrt{\left(\frac{(2n+2+A_K)\pi}{Lm} - \frac{1}{Lm}[\delta(2\frac{(2n+A_K)\pi}{Lm})+2\pi]\right)^2+1},\end{aligned}$$

where we have shifted $\frac{2\pi}{Lm}$ from the phase term to the first term. The reason for this is that now one can consistently expand the root in the following integral since $\frac{z[\delta+2\pi]}{Lm}$ is even for $z\to\infty$ a small quantity. Without this shift by 2π the approximation would break down and result in a divergence.

Putting all together (156),(167),(315) and inserting in the Euler-MacLaurin formula one gets with the variable transformation $z = \frac{(2n+2+A_K)\pi}{Lm}$:

$$\begin{aligned}M_K = \;& M_{cl} + \hbar m\frac{\sqrt{3}}{4} + \frac{\hbar m}{2}(A_K-1) - \frac{\hbar m}{2} + \hbar m\left(\int_{\frac{(2+A_K)\pi}{Lm}}^{\Lambda+\frac{A_K\pi}{Lm}} - \int_{\frac{2\pi}{Lm}}^{\Lambda}\right)\frac{dz}{2\pi}Lm\sqrt{z^2+1}\\ &-\frac{\hbar m}{2}A_K\sqrt{\Lambda^2+1} - \hbar m\left(\int_0^\Lambda\frac{dz}{2\pi}\frac{z[\delta(2z)+2\pi]}{\sqrt{z^2+1}} + 3\int_0^\Lambda\frac{dz}{2\pi}\frac{1}{\sqrt{z^2+1}}\right) + O(\frac{1}{L}) + O(\hbar^2)\end{aligned}$$

In the second line we have already carried out $L\to\infty$ since the integrand is independent of L. The surface terms of Euler-MacLaurin formula have canceled each other. Doing the integrals and taking the limit $L\to\infty$ one obtains

$$\begin{aligned}M_K = \;& M_{cl} + \hbar m\frac{\sqrt{3}}{4} + \frac{\hbar m}{2}\left(A_K - 1 - 1 - A_K + A_K\sqrt{\Lambda^2+1} - A_K\sqrt{\Lambda^2+1}\right)\\ &-\hbar m\frac{1}{2\pi}(3-2\pi) - \hbar m\frac{1}{2\sqrt{3}}\end{aligned}$$

So again the divergences cancel each other by correct mode counting, and one obtains, independently of the BC-combination, for the kink mass

$$M_K = M_{cl} + \hbar m\left(\frac{1}{4\sqrt{3}} - \frac{3}{2\pi}\right) + O(\hbar^2)$$

Similarly, other choices of the branch cut positions lead to exactly the same result. This is as it should be, since the choice of a certain branch cut position is completely unphysical and a purely mathematical convention. So no calculations and results should depend on it.

4.3 Hilbert space of the soliton sector

Since we now know the spectrum of the soliton sector (up to order $\hbar$) we can consider the particle content of this sector. The energy spectrum in the kink sector is of the form[33]

$$E[\{\nu_l\}] = M_K + \omega_d \nu_d + \sum_n \omega_n \nu_n,$$

where, in contrast to (108), M_K is the one-loop quantum mass of the kink. Thus these are the excitations of the kink at rest. For the ϕ^4 there also exists a discrete mode (the zero mode is not included). This is the spectrum if the kink is at rest. Like in the vacuum sector, to each mode $\{d, n\}$ and their multiple excitation a state in the Hilbert (Fock) space is associated. Single excitations ($\nu_n = 1$) correspond to the fundamental quanta (in the presence of the kink) or even to new particles ($\nu_d = 1$ or the kink itself). This states are different from the vacuum states, as we will show. Thus in addition to the vacuum Hilbert space (vacuum and multi meson states) there exists a kink sector Hilbert space. In the vacuum sector, besides the vacuum state $|0\rangle$ only continuum states (plane wave eigen-functions) $|k_1, \dots k_n\rangle$ exists, which correspond to the fundamental quanta of the theory. To perform localized (normalizable) particle states one has to built up wave packets of these states. In the kink sector besides the continuum modes there exists the kink, which has a localized energy density, and also a discrete mode (for ϕ^4), which is normalizable. Thus the *kink sector Hilbert space* consists of the following elements (particle states)

1. The lowest state is the kink particle $|P\rangle$ with the momentum P and the energy $E = \sqrt{P^2 + M_K^2}$ (see (112)).
2. The excited state of the kink $|P^*\rangle$ (only for ϕ^4) of momentum P and energy $E = \sqrt{P^2 + (M_K + \omega_d)^2}$.
3. The scattering states $|P, k_1, \dots, k_n\rangle$ consisting of the kink particle and n mesons scattering of the kink with asymptotic momenta $P, k_1, \dots k_n$.
4. The scattering states $|P^*, k_1, \dots, k_n\rangle$ consisting of the excited kink and n mesons of asymptotic momenta $P, k_1, \dots k_n$.

For the state $|P^*\rangle$ one discrete mode $\omega_1 = m\frac{\sqrt{3}}{2}$ is excited (see appendix). Higher excitations of this mode ($\nu_1 > 1$) will be unstable against decay into kink and meson since its energy $\nu_1\omega_1 = \nu_1\frac{\sqrt{3}}{2}$ lie for $\nu_1 > 1$ above the meson mass m. Note that with the zero mode no new state is associated, since it reflects only the collective motion of the kink and is therefore contained in the energy-momentum relation of the kink in point 1.

We have already seen that the kinks are stable under small perturbations (all eigen-values of the stability equation are positive) and that the stability or the existence of non-dissipative solutions, respectively, is connected with the existence of a topological conservation law (there also exist localized finite energy solutions whose stability arises from ordinary conservation laws. These solutions are necessarily time-dependent, like the sine-Gordon breather [39]). We have also seen, that the different topological sectors are not connected. We show that this is

[33] In this section we use units so that $\hbar = 1$.

also true in the quantized theory. In terms of Hilbert space this is expressed by the following two postulates [38], [3]:

I. The kink sector Hilbert space is *orthogonal* to the vacuum Hilbert space, i.e. for all amplitudes

$$\langle kink\ sector | vacuum\ sector \rangle = 0.$$

This can be shown as follows: The topological charge operator[34]

$$\mathcal{Q} = \int dx \mathcal{J}_0 = c \left[\hat{\phi}(\infty, t) - \hat{\phi}(-\infty, t) \right], \tag{179}$$

is conserved in time since $\mathcal{J}_0$ is the zero component of the conserved current

$$\mathcal{J}^\mu = \varepsilon^{\mu\nu} \partial_\nu \hat{\phi} \quad \text{with} \quad \partial_\mu \mathcal{J}^\mu = 0.$$

Since $\hat{\phi}$ is hermitian also $\mathcal{Q}$ is a hermitian operator and thus its eigen-states are orthogonal. In addition $\mathcal{Q}$ is conserved in time and translationally invariant (see (179)) , i.e. it commutes with the energy-momentum operator $\mathcal{P}_\mu$. Thus, independent of the considered sector there exists a basis in the Hilbert space so that $\mathcal{P}_\mu$ and $\mathcal{Q}$ have common eigenstates. Therefore the eigenvalues of $\mathcal{Q}$ are "good" quantum numbers and each state in the Hilbert space can be characterized by them. Thus the existence of conserved topological charge $\mathcal{Q}$ has, due to the existence of a conserved current, analogous consequences as usual Noether charges, following from continuous symmetries. Now the action of the field operator on vacuum states are of the form

$$\hat{\phi}(x,t) \left| k_1, \ldots k_n \right\rangle = \phi_V + \hat{\eta}_V(x,t) \left| k_1, \ldots k_n \right\rangle \underset{x \to \pm\infty}{\longrightarrow} \phi_V + \text{rapidly oscillating terms.}$$

The rapidly oscillating terms do not contribute to the charge $\mathcal{Q}$ (see (29)), so that for all states of the vacuum sector the topological quantum number $Q = 0$. Whereas the action on kink sector states is of the form[38]

$$\hat{\phi}(x,t) \left| P, k_1, \ldots k_n \right\rangle = \phi_K(x,t) + \hat{\eta}_K(x,t) \left| P, k_1, \ldots k_n \right\rangle \underset{x \to \pm\infty}{\longrightarrow} \phi_K(\pm\infty, t) + \text{rapidly oscillating terms.}$$

The rapidly oscillating terms in the kink sector are the same as in the vacuum sector, up to linear combinations (see appendix), and therefore also do not contribute to the charge. But the kink function ϕ_K gives a non-trivial contribution, so that all states in the kink sector have the topological charge $Q = 1$. Since states with different topological charge are orthonormal one obtains for the amplitudes

$$\langle kink\ sector | vacuum\ sector \rangle = \langle Q = 1 | Q = 0 \rangle = 0.$$

Since $\mathcal{Q}$ is a conserved operator the kink and the vacuum sector are not only orthogonal but cannot evolve into one another. Thus, despite all kink sector states, built around the *local* minimum ϕ_K, have higher energy (in the weak coupling limit, since $M_{cl} = O(\frac{1}{\lambda})$) than any vacuum sector state, they do not decay into the vacuum vector states, built around the lower minimum, as expected *a priori*, purely from energetics. In the quantized theory this is apparently due to the existence of the conserved topological charge.

[34] c is a normalization constant, see (26).

II. The different topological Hilbert space sectors are not connected by any localized operator. Consider any localized physical observable $\mathcal{A}(t)$. That is, let

$$\mathcal{A}(t) = \int dx \hat{a}(x,t) dt,$$

where $\hat{a}(x,t)$ is a local function of the field and its derivatives with finite spatial support at any given time t. Then the equal time commutator

$$[\mathcal{A}(t), \mathcal{Q}(t)] = \lim_{L\to\infty} c \Big\{ [\mathcal{A}(t), \hat{\phi}(L,t)] - [\mathcal{A}(t), \hat{\phi}(-L,t)] \Big\} = 0$$

because of the causality condition, i.e. that all commutators of space-like separated operators vanish. Thus, any such operator $\mathcal{A}$ cannot connect sectors with different topological charges Q. This suggests that Q is something like a super-selection quantum number, separating the kink sector from the vacuum sector.

4.4 Continuum calculation (EMC) for bosonic kinks

In [37] it was shown that the widely used (see references in [37]) common strict energy-momentum-cutoff regularization (EMC) leads to results for the kink masses which differ from that obtained by a mode regularization. In [37] the EMC were identified as incorrect by comparing the calculated masses with exact results known for the SG breather solution. In recent works ([32],[33]) a remedy has been suggested using an analogy to the Casimir effect, which in any physically realistic situation has a natural UV cutoff. There are however several reasons why we think that this solution is not satisfying: (i) In a discretized calculation this would impose the (as we have seen) unnecessary requirement of identical boundary conditions in the topologically distinct sectors; (ii) It also depends in a crucial way on the position of the branch cut of the scattering phase which is completely unphysical. The procedure works only for discontinuous phases which go asymptotically to zero. However, as we shall see, in the supersymmetric case this is impossible because of the presence of half-bound states.

In our opinion, the deeper question behind this whole issue which has to be solved is how to regularize/renormalize two different topological sectors (sectors with trivial and non-trivial background) in a consistent way. Therefore one needs a principle that tells one how to regularize two sectors in the "same way" so that one can compare them in a consistent manner (the prescription of a common energy cutoff evidently does not achieve this). The principle must determine in particular how to relate regularization parameters (cutoffs) in the two sectors. In a mode regularization scheme this is done by mode counting, i.e by the requirement that both sectors below a certain energy/momentum have the same dimension in field-configuration space. In a continuum calculation, the dimension can be measured by the spectral density. But these spectral densities are the quantities to be determined. So one needs an independent principle which relates the cutoff in the kink sector to the cutoff in the vacuum sector. We show that the requirement that the regularized units in the two sectors can be matched will achieve this.

Required accuracy

In the discrete case one has two approximation parameters L and Λ which can be used in calculations. In the continuum calculation the only regularization parameter is the cutoff Λ.

This cutoff is fixed in the vacuum sector and by renormalization, i.e. the counter-terms defined by renormalization conditions. The important thing is, that the theory should be renormalized only once to be consistent. Therefore all other regularization quantities especially in the nontrivial sector must be given by Λ in a unique way. These relations and other approximations must be accurate to sufficiently high orders, so that no finite contributions are lost and finite errors survive. For calculating the energy they are given as follows:

Cutoffs: The integration boundaries in the soliton sector, must be correct up to order $O(\frac{1}{\Lambda^2})$ since the mode energies, i.e the integrand are of order $O(\Lambda)$ for high momenta and therefore orders $O(\frac{1}{\Lambda})$ in the cutoff give finite contributions even in the limit $\Lambda \to \infty$.

Spectral densities: The spectral density $\rho = \rho_{kink} - \rho_{vac}$ measures the (difference of the) number of states in the continuum. Since in the kink sector, there are also discrete states, its integral should give the negative number of these discrete states. The integral of the spectral density must give the correct number of discrete states up to order $O(\frac{1}{\Lambda^2})$ so that the error in the number of continuum states does not contribute to the energy. An error of order $O(\frac{1}{\Lambda})$ would result in a finite error in the energy since the wrongly counted modes are multiplied with the mode energies which are at the high end of order $O(\Lambda)$. This is the analogue of mode counting in the discrete case where it is of course much simpler.

4.4.1 Vacuum

To calculate the ground state energy of the vacuum fluctuations one has to path-integrate the the quadratic part of the action

$$\frac{1}{2\hbar}\int\limits_{T\mathrm{x}\mathbb{R}} dtdx\ \eta(x,t)\frac{\delta^2 S}{\delta\phi^2}|_{\phi_V}\eta(x,t) \tag{180}$$

Therefore one diagonalizes the operator in (180) as follows ($z = \frac{mx}{l}$, $l = 1, 2$ for SG, ϕ^4)

$$(-\partial_z^2 + l^2)\xi_V(k,z) = \omega_V^2(k)\xi_V(k,z) \tag{181}$$

$$\xi_V(k,z) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{ikz} \quad \omega^2(k) = k^2 + l^2 \tag{182}$$

The energy of a mode is then given as

$$E_V(k) = \frac{\hbar m}{2l}\omega(k) = \frac{\hbar m}{2l}\sqrt{k^2 + l^2} \tag{183}$$

To diagonalize the quadratic part of the action one has to expand the quantum fluctuations $\eta(z,t)$ according to the eigen-functions (182). Here the cutoff regularization takes place as follows

$$\eta(z,t) = \int dk\Theta(\Lambda - |k|)\xi_V(k,z)\alpha(k,t) \tag{184}$$

After spatial integration in (180) the quadratic part of the action is a continuous set of harmonic oscillators $\alpha(k,t)$ with the energies (183). The continuous set is strictly cut off at the momentum $|k| = \Lambda$. This cutoff characterizes the set of fluctuations which are considered in the path integration (159). In this sense regularization means to restrict the path integration on a subset of the continuous functions

$$PID = \{C[\mathbb{R}\mathrm{x}T] \mid \eta(x,t') = \eta(x,t'') = \eta_a(x)\} \longrightarrow PID_{Reg}$$

The regularized subset of the considered functions is characterized by the eigen-functions which are taken into account in (184). The subset PID_{Reg} is obtained by the action of a projection operator in the whole space PID which is the identity in PID_{reg} and we therefore call the *regularized unit*:

$$\hat{\delta}_V(z-z') = \int dk\Theta(\Lambda-|k|)\xi_V^*(k,z')\xi_V(k,z) = \frac{1}{2\pi}\int_{-\Lambda}^{\Lambda} dk e^{ik(z-z')} = \frac{1}{\pi}\frac{\sin[\Lambda(z-z')]}{(z-z')} \quad (185)$$

The proof that this is a projector and it is the unit in PID_{Reg} is straightforward

$$\hat{\delta}_V^2 = \int dy \hat{\delta}_V(z-y)\hat{\delta}(y-z') = \hat{\delta}_V(z-z') = \hat{\delta}_V$$
$$\hat{\delta}_V\eta = \int dz' \hat{\delta}_V(z-z')\eta(z',t) = \eta,$$

where in the second line we have inserted (184) for η. There are two interesting limits of (185)

$$\lim_{\Lambda\to\infty}\hat{\delta}_V(z-z') = \delta(z-z') \quad (186)$$
$$\lim_{z'\to z}\hat{\delta}_V(z-z') = \frac{\Lambda}{\pi} \quad (187)$$

The first line is obvious since in this limit (185) is a representation of the Dirac delta-distribution; the second limit provides the diagonal elements which will be needed later.

4.4.2 Kink sector

In the nontrivial sector the quadratic part of the action is given by

$$\frac{1}{2\hbar}\int\limits_{T\mathrm{x}\mathbb{R}} dtdx\ \eta(x,t)\frac{\delta^2 S}{\delta\phi^2}|_{\phi_K}\eta(x,t)$$

The operator $\frac{\delta^2 S}{\delta\phi^2}|_{\phi_K}$which must be diagonalized is the stability operator. Its spectrum is given in the appendix (314, 315). The mode energies are given as in the vacuum (182) by

$$E_K(k) = \frac{\hbar m}{2l}\omega_K(k) = \frac{\hbar m}{2l}\sqrt{k^2+l^2} \quad (188)$$

The difference to the vacuum is given by the different eigen and continuum states which will lead to a spectral density which differs from that in the vacuum sector. The spectral density for the continuous spectrum of a differential operator relative to its "free" part, i. e. the associated vacuum operator is given by the diagonal elements of their density matrices ([42],[43]):

$$\rho(k) = \int_{-\infty}^{\infty} dz\,[\xi_K^*(k,z)\xi_K(k,z) - \xi_V^*(k,z)\xi_V(k,z)] \quad (189)$$

This spectral density without cutoff regularization is usually used for zeta-function regularization [40]. Setting up the same strict cutoff in both sectors, i.e. multiplying (189) with a common step-function, one obtains the same spectral density as in [37]

$$\rho_{com}(k) = \Theta(\Lambda-|k|)\frac{\delta'(k)}{2\pi} \quad (190)$$

Usual this spectral density is calculated by starting from a mode regularization (see [37] and references therein). Also in [32] this detour has been taken. This is the reason why one gets problems with the BC and the branch cut position, since the modifications in [32] only lead to a certain mode number cutoff scheme (certain BC and branch cut position). Other combinations cannot be produced by the Casimir trick.

In the next sections we calculate the correction to (190) appropriate for an EMC scheme. First we construct the analogue to the regularized unit (185) in the kink sector assuming a different cutoff Λ_K. The requirement of a consistent regularization will define Λ_K as a function of the given cutoff Λ, defined by the vacuum. The consistency of the regularization is given by the requirement that the path integration over quantum fluctuations must be restricted in *both* sectors to the *same* subset PID_{reg} ("regularized path integration domain"). To make this notion more concrete is the subject of the following sections.

4.4.3 Sine-Gordon

For the sine-Gordon model the normalized eigen states are given by

$$\xi_0 = \frac{1}{\sqrt{2}} \frac{1}{\cosh z} \tag{191}$$

$$\xi(k,z) = \frac{e^{ikz}}{\sqrt{2\pi}} \frac{\tanh z - ik}{\sqrt{k^2+1}} \tag{192}$$

The regularized unit is given as

$$\hat{\delta}_K(z,z') = \frac{1}{2} \frac{1}{\cosh z \cosh z'} + \int dk \Theta(\Lambda_K - |k|) \frac{e^{ik(z-z')}}{2\pi} \frac{(\tanh z' + ik)(\tanh z - ik)}{k^2+1} \tag{193}$$

Because the eigen-values (314) are symmetric (degenerated) in k we made symmetric ansatz for the cutoff Λ_K, i.e. we choose the same for positive and negative momenta. For the calculation of the spectral density we only need the diagonal elements of the density matrix $\xi^*(k',z')\xi(k,z)$. Therefore it is sufficient to know Λ_K at the point $z' \to z$:

$$\lim_{z' \to z} \hat{\delta}_K(z,z') = \frac{1}{2\cosh^2 z} + \int_{-\Lambda_K}^{\Lambda_K} \frac{dk}{2\pi} \frac{\tanh^2 z + k^2}{k^2+1} \tag{194}$$

The integration is easily performed and one obtains

$$\hat{\delta}_K(z,z) = \frac{\Lambda_K}{\pi} + \frac{1}{\cosh^2 z} \left(\frac{\pi - 2\arctan \Lambda_K}{2\pi} \right) = \frac{\Lambda_K}{\pi} + \frac{1}{2\pi} \frac{1}{\cosh^2 z} \delta(\Lambda_K) \tag{195}$$

It is interesting that the scattering phase arises here. In the discrete calculation the scattering phase δ comes in by setting up boundary conditions on the asymptotic states. And indeed as shown in [37] the shift in the cutoff is connected with the scattering phase. Note that the function $\delta(\Lambda_K) = \pi - 2\arctan \Lambda_K$ is uniquely given by the integration (194). Thus there is no ambiguity in choosing a certain branch cut position.

To determine the cutoff Λ_K we require that the regularized subset PID_{Reg} of paths in the kink sector is in a certain sense the same as in the vacuum. To establish this we require that the two projectors $\hat{\delta}_V$ and $\hat{\delta}_K$ must coincide (in the sense of distributions). Since for the

spectral density we need only the diagonal elements it is sufficient to use (195) and (185). The requirement of the equivalence of the off diagonal elements can be understood as an equivalence of the two operators in higher orders of $O(\frac{1}{\Lambda})$, which however is not needed for our purpose. Thus we get

$$\hat{\delta}_K \stackrel{!}{=} \hat{\delta}_V \Longrightarrow \Lambda_K + \frac{1}{\cosh z^2}\frac{\delta(\Lambda_K)}{2} \stackrel{!}{=} \Lambda \tag{196}$$

This is an implicitly given function for Λ_K since the cutoff Λ is given and defined by the vacuum and renormalization. Therefore we have to solve (196) for Λ_K. This we do by application of the Banach fixed point theorem in an iteration up to sufficient order in Λ. After the first iteration we are at sufficient order

$$\Lambda_K = \Lambda - \frac{1}{2\cosh^2 z}\delta(\Lambda) + O([\frac{1}{2\cosh^2 z}]^2, \frac{1}{\Lambda^3}) \tag{197}$$

In (197) both factors $\frac{1}{2\cosh^2 z}$ and $\frac{1}{\Lambda}$ are smaller than one (if $\Lambda > 1$) and therefore one has independently of z a contraction so that the Banach fixed point theorem is applicable and (197) is a reasonable approximation. In the limit $\Lambda \to \infty$ the unit defined in the kink sector (193) and the kink cutoff Λ_K (197) converge to the vacuum quantities as

$$\begin{aligned}
&\lim_{\Lambda\to\infty} \Lambda_K = \Lambda + O(\frac{1}{\Lambda}) \\
&\lim_{\Lambda\to\infty} \hat{\delta}_K(z,z') \to \delta(z-z')
\end{aligned}$$

Thus the kink cutoff Λ_K approaches Λ as $\frac{1}{\Lambda}$ and therefore this difference can never be neglected (see section 4.4). The second relation is nothing else than the completeness relation for the spectrum of the self adjoint stability operator. Relation (196) is the analogue of the requirement of equal number of modes in the discrete case. This ensures that the considered subset of paths has in both cases the same dimension.

We are now in the position to calculate the cutoff-regularized spectral densities. First we define some notational abbreviation to keep the calculations readable.

$$\begin{aligned}
&\Theta_\Lambda(|k|) := \Theta(\Lambda - |k|) \\
&\delta := \delta(\Lambda) \\
&\Theta_{\Lambda_K}(|k|) := \Theta\left(\Lambda - \frac{1}{\cosh^2 z}\frac{\delta}{2} - |k|\right) \\
&\Theta_{(\Lambda-\frac{1}{2}\delta,\Lambda)}(|k|) = 1 \text{ for } k \in (\Lambda - \frac{1}{2}\delta, \Lambda)\ ;\ \text{else } 0
\end{aligned}$$

Respecting the different cutoffs one gets the spectral density in an analogous way to (189) as

$$\begin{aligned}
\rho(k) &= \int dz\, [\Theta_{\Lambda_K}(|k|)\xi_K^*\xi_K - \Theta_\Lambda(|k|)\xi_V^*\xi_V]_{(k,z)} && (198)\\
&= \Theta_\Lambda(|k|)\int dz\, [\xi_K^*\xi_K - \xi_V^*\xi_V] + \int dz\, [\Theta_{\Lambda_K}(|k|) - \Theta_\Lambda(|k|)]\, \xi_K^*\xi_K && (199)\\
&=: \rho_{com}(k) + \Delta\rho(k), && (200)
\end{aligned}$$

where we have split the spectral density into the conventional part obtained by a strict common cut off ρ_{com} and the correction $\Delta\rho$. The integration for ρ_{com} is elementary and give the well

known result

$$\rho_{com}(k) = \Theta_\Lambda(|k|) \int \frac{dz}{2\pi} [\frac{\tanh^2 z + k^2}{k^2+1} - 1] = \tag{201}$$

$$= \Theta_\Lambda(|k|) \frac{1}{2\pi} \frac{-2}{k^2+1} = \Theta_\Lambda(|k|) \frac{1}{2\pi} \delta'(k). \tag{202}$$

To show that the spectral density ρ_{com} is not correct up at the order of interest we integrate (202) over k, which gives

$$\int dk \rho_{com} = -\frac{2}{\pi} \arctan \Lambda = -1 + \frac{2}{\pi} \frac{1}{\Lambda} + O(\frac{1}{\Lambda^2}) \tag{203}$$

In the last step we have expanded $\arctan \Lambda$ around $\Lambda = \infty$. From (203) one can see that the conventional spectral density gives the correct (negative) number of discrete states (1 in the sine-Gordon model) only up to errors of order $O(\frac{1}{\Lambda})$. Next we calculate the correction $\Delta\rho$ to the spectral density ρ_{com}. It is symmetric in z and therefore we can restrict the considerations to positive z with a factor two. With (192), (200) one obtains

$$\Delta\rho(k) = \frac{1}{\pi} \int_0^\infty dz \left[\Theta_{\Lambda_K}(|k|) - \Theta_\Lambda(|k|)\right] \frac{\tanh^2 z + k^2}{k^2+1} \tag{204}$$

With a variable transformation $x = \cosh z$ (204) can be written as

$$\Delta\rho(k) = \frac{1}{\pi} \frac{1}{k^2+1} \int_1^\infty dx \left[\Theta(\Lambda - \frac{1}{2x^2}\delta - |k|) - \Theta(\Lambda - |k|)\right] \left(\frac{\sqrt{x^2-1}}{x^2} + \frac{k^2}{\sqrt{x^2-1}}\right) \tag{205}$$

Because of the step functions in (205) the integral is only unequal zero if $|k| < \Lambda$. In this case the first bracket in (205) takes the values

$$0 \ldots \Lambda - |k| - \frac{\delta}{2x^2} > 0 \tag{206}$$

$$-1 \ldots \Lambda - |k| - \frac{\delta}{2x^2} < 0 \tag{207}$$

Therefore the integration in (205) is restricted to values

$$x^2 < \frac{\delta(\Lambda)}{2(\Lambda - |k|)} \tag{208}$$

The smallest possible value of x is the lower integration boundary in (205), i.e. $x = 1$. This leads to a constraint for the possible k values (else $\Delta\rho = 0$)

$$1 < \frac{\delta(\Lambda)}{2(\Lambda - |k|)} \Rightarrow |k| > \Lambda - \frac{\delta(\Lambda)}{2}$$

Therefore the correction is only nonzero if $|k| \in (\Lambda - \frac{\delta}{\Lambda}, \Lambda)$. With (208) and (207) we get

$$\Delta\rho(k) = -\Theta_{(\Lambda - \frac{\delta}{2}, \Lambda)}(|k|) \frac{1}{\pi} \frac{1}{k^2+1} \int_1^{\sqrt{\frac{\delta}{2(\Lambda-|k|)}}} dx \left(\frac{\sqrt{x^2-1}}{x^2} + \frac{k^2}{\sqrt{x^2-1}}\right) \tag{209}$$

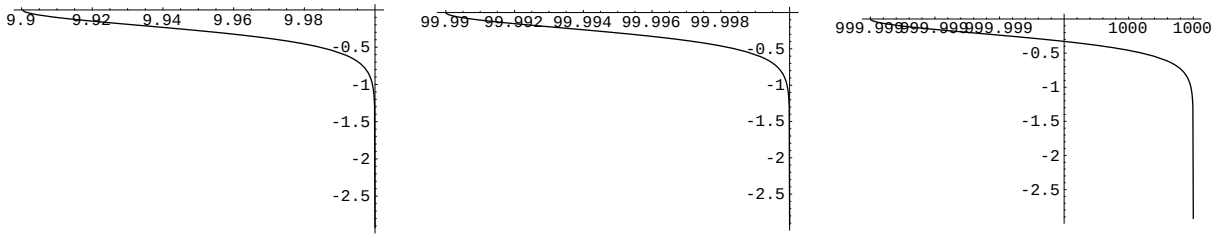


Figure 12: The spectral density correction $\Delta\rho(k)$ for the values $\Lambda = 10$, $\Lambda = 100$ and $\Lambda = 1000$ of the dimensionless vacuum cutoff.

The integration in (209) can be carried out exactly. The final result for the correction is using the abbreviation $\alpha_\Lambda(k) := \frac{\delta(\Lambda)}{2(\Lambda - |k|)}$:

$$\Delta\rho(k) = -\Theta_{(\Lambda-\frac{\delta}{2},\Lambda)}(|k|)\frac{1}{\pi}\left(\ln[\sqrt{\alpha_\Lambda(k)} + \sqrt{\alpha_\Lambda(k)-1}] - \frac{1}{k^2+1}\sqrt{1-\frac{1}{\alpha_\Lambda(k)}}\right) \tag{210}$$

Although this expression is rather complicated it is not exact. When solving the implicit given function for Λ_K (196) after Λ we had to approximate the solution up to order $O(\frac{1}{\Lambda^3})$ (197). The correction is a smooth function of k and its graph is shown in fig. 12 for different cutoffs Λ. Let us now verify that the approximate solution (197) is sufficiently accurate. As a test we show that the integral of the spectral density gives the correct number of bound states at least to the sufficient order $O(\frac{1}{\Lambda^2})$:

$$\int dk\Delta\rho(k) = 2\int_0^\infty dk\Delta\rho(k) \tag{211}$$

$$= -\frac{2}{\pi}\int_{\Lambda-\frac{\delta}{2}}^{\Lambda} dk\left(\ln[\sqrt{\alpha_\Lambda(k)} + \sqrt{\alpha_\Lambda(k)-1}] - \frac{1}{k^2+1}\sqrt{1-\frac{1}{\alpha_\Lambda(k)}}\right) \tag{212}$$

The logarithm can be integrated exactly by a variable transformation $z = \frac{2(\Lambda-k)}{\delta(\Lambda)}$ and gives

$$-\frac{\delta(\Lambda)}{\pi}\int_0^1 dz[\ln(1+\sqrt{1-z}) - \ln\sqrt{z}] = -\frac{\delta(\Lambda)}{\pi} \tag{213}$$

The second term in (212) we can only estimate as follows

$$\frac{2}{\pi}\int_{\Lambda-\frac{\delta}{2}}^{\Lambda} dk\frac{1}{k^2+1}\sqrt{1-\frac{1}{\alpha_\Lambda(k)}} < \frac{2}{\pi}\int_{\Lambda-\frac{\delta}{2}}^{\Lambda} dk\frac{1}{k^2+1} = \tag{214}$$

$$= \frac{2}{\pi}[\arctan\Lambda - \arctan(\Lambda - \frac{\delta}{2})] = \frac{2}{\pi}\frac{1}{\Lambda^3} + O(\frac{1}{\Lambda^5}) \tag{215}$$

Therefore we can neglect this contribution to the spectral density test since it is of order $O(\frac{1}{\Lambda^3})$ and so does not contribute as discussed in section 4.4. Thus we have for the complete spectral density integral the following result (202),(213)

$$\int dk(\rho_{com} + \Delta\rho) = -\frac{2}{\pi}\arctan\Lambda - \frac{1}{\pi}(\pi - 2\arctan\Lambda) + O(\frac{1}{\Lambda^3}) = -1 + O(\frac{1}{\Lambda^3}) \tag{216}$$

Thus the integral of the corrected spectral density gives the correct number of discrete states up to the sufficient order $O(\frac{1}{\Lambda^3})$. The remaining error is only due to our effort to express Λ_K through the the regularization/renormalization defining vacuum cutoff Λ. Therefore we had to solve (196) in a reasonable approximation. Setting up Λ_K as the fundamental cutoff and expressing Λ as a function of it one obtains a slightly different correction to the spectral density ρ_{com} ($\rho = \rho_{com} - \tilde{\Delta}\rho(k)$):

$$\tilde{\Delta}\rho(k) = \int dz \left[\Theta(\Lambda_K + \frac{1}{2\cosh^2 z}\delta(\Lambda_K) - |k|) - \Theta(\Lambda_K - |k|)\right] \xi_V^*(k,z)\xi_V(k,z)$$

An analogous (and simpler) calculation as above gives

$$\tilde{\Delta}\rho(k) = \frac{1}{\pi}\Theta_{(\Lambda_K, \Lambda_K + \frac{1}{2}\delta(\Lambda_K))}(|k|) \ln\left[\sqrt{\frac{\delta(\Lambda_K)}{2(|k| - \Lambda_K}} + \sqrt{\frac{\delta(\Lambda_K)}{2(|k| - \Lambda_K)} - 1}\right]$$

The integral of the spectral density gives now

$$\int dk\rho(k) = \frac{1}{2\pi}(-2\arctan\Lambda_K) - 1 + \frac{1}{\pi}\arctan\Lambda_K = -1$$

This is indeed the exact (negative) number of discrete states. Thus, a complete matching of the diagonal elements of the regularized units gives even an exact result for the sum rule to be satisfied by the spectral density.

4.4.4 Quantum mass of the kink

The correction $\Delta\rho$ results in a additional contribution to quantum mass of the kink compared to the mass calculated with the spectral density with a strict common cutoff ρ_{com}. With (183) and (188) one obtains

$$\Delta M = \frac{\hbar m}{2}\int dk\sqrt{k^2+1}\Delta\rho(k) \tag{217}$$

$$= \hbar m(\frac{-1}{\pi})\int_{\Lambda-\frac{\delta}{2}}^{\Lambda} dk\sqrt{k^2+1}\left(\ln[\sqrt{\alpha_\Lambda(k)} + \sqrt{\alpha_\Lambda(k)-1}] - \frac{1}{k^2+1}\sqrt{1-\frac{1}{\alpha_\Lambda(k)}}\right) \tag{218}$$

The second term in (218) can be estimated as follows

$$\frac{\hbar m}{\pi}\int_{\Lambda-\frac{\delta}{2}}^{\Lambda} dk\frac{1}{\sqrt{k^2+1}}\sqrt{1-\frac{1}{\alpha_\Lambda(k)}} < \frac{\hbar m}{\pi}\int_{\Lambda-\frac{\delta}{2}}^{\Lambda} dk\frac{1}{\sqrt{k^2+1}}$$

$$= \frac{\hbar m}{\pi}(\text{arsinh}\Lambda - \text{arsinh}(\Lambda - \frac{\delta(\Lambda)}{2}) \stackrel{\Lambda\to\infty}{\longrightarrow} 0.$$

To integrate the logarithm we transform to the variable $z = \frac{2(\Lambda-k)}{\delta}$. This gives

$$-\frac{\hbar m}{\pi}\frac{\delta(\Lambda)}{2}\int_0^1 dz\sqrt{(\Lambda - \frac{\delta}{2}z)^2+1}\left(\ln(1+\sqrt{1-z}) - \ln\sqrt{z}\right)$$

Expanding the root and with $\int_0^1 dz(\ln(1+\sqrt{1-z}) - \ln\sqrt{z}) = 1$ one obtains for the correction

$$\Delta M = -\hbar m\frac{1}{2\pi}\delta(\Lambda)\Lambda + O(\frac{1}{\Lambda^2}) \stackrel{\Lambda\to\infty}{\longrightarrow} -\frac{\hbar m}{\pi} \tag{219}$$

This is exactly the missing contribution in the strict common cutoff calculation [37].

4.4.5 The large Λ limit and comparison with the results of [32]

We want to demonstrate that the correction to the spectral density obtained by [32] which is sharply located at $|k| = \Lambda$ can be obtained by a large Λ limit keeping the necessary orders. Therefore one must respect that the correction $\Delta\rho$ (210) for varying Λ is a sequence of distributions acting on "test-functions" like the mode energy (183)is one. So we investigate the action of $\Delta\rho$ on test-functions in the large Λ limit. As test-functions we use smooth symmetric functions $\varphi(k) = \varphi(-k)$ like the mode energy (183) is one, which grow at most linearly with k and for $|k| > \Lambda$ we can assume that they vanish fast enough to be a test-function, since we are considering large but still finite Λ. This symmetry is not really necessary but so we don't have to treat the $k < 0$-domain separately. Thus we consider the following expression for large Λ:

$$l_{\Delta\rho}(\varphi) = \int dk \Delta\rho(k)\varphi(k) = 2\int_0^\Lambda dk \varphi(k)\Delta\rho(k) \tag{220}$$

First we calculate the logarithmic part of (220). With (210) and a variable transformation $z = \frac{2(\Lambda-k)}{\delta}$ we get

$$-\frac{\delta(\Lambda)}{\pi}\int_0^1 dz \varphi(\Lambda - \frac{\delta}{2}z)[\ln(1+\sqrt{1-z}) - \ln\sqrt{z}] \tag{221}$$

$$= -\frac{\delta(\Lambda)}{\pi}\int_0^1 dz [\varphi(\Lambda) - \varphi'(\Lambda)\frac{\delta(\Lambda)}{2}z + O(\delta^2, z^2)][\ln(1+\sqrt{1-z}) - \ln\sqrt{z}] \tag{222}$$

$$= -\frac{\delta(\Lambda)}{\pi}[\varphi(\Lambda) + \frac{1}{3}\varphi'(\Lambda)\frac{\delta(\Lambda)}{2} + O(\frac{1}{\Lambda^2})] = -\frac{\delta(\Lambda)}{\pi}\varphi(\Lambda) + O(\frac{1}{\Lambda^2}). \tag{223}$$

The action of the second part of (210) we estimate as follows

$$\frac{2}{\pi}\int_{\Lambda-\frac{\delta}{2}}^\Lambda \varphi(k)\frac{1}{k^2+1}\sqrt{1-\frac{2(\Lambda-k)}{\delta}} < \frac{2}{\pi}|\varphi(\Lambda)|\int_{\Lambda-\frac{\delta}{2}}^\Lambda \frac{1}{k^2+1} \tag{224}$$

$$= \frac{2}{\pi}|\varphi(\Lambda)|\frac{1}{\Lambda^3} + O(\frac{1}{\Lambda^6}) \tag{225}$$

Therefore get for (220) with (223),(225)

$$l_{\Delta\rho}(\varphi) = -\frac{\delta(\Lambda)}{\pi}\varphi(\Lambda) + O(\frac{1}{\Lambda^2})$$

Thus in the large Λ limit the distribution $\Delta\rho$ approaches

$$\Delta\rho(k) \to -\frac{\delta(\Lambda)}{\pi}\delta^D(\Lambda - |k|) = -\frac{1}{\pi}\delta(k)\delta^D(\Lambda - |k|)$$

This is exactly the additional term in the spectral density obtained by [32] using an analogy to the Casimir effect. Therefore we understand the results of [32] as a large Λ limit of the smooth correction $\Delta\rho$ (210). But the calculation in [32] suffers from the problem that it works only for certain branch cut positions of the scattering phase. Other branch cut positions lead to a divergent result for the kink mass. This ambiguity cannot be fixed without the use of mode number cutoff considerations. In our case the occurrence of the function $\pi - 2\arctan\Lambda$, which equals the scattering phase $\delta(\Lambda)$ with a certain branch cut position, in (195) is completely unique. It simply comes from a uniquely defined integral.

4.4.6 Robustness of the procedure

We now investigate the z-dependence and the stability of the relation between the cutoffs Λ_K, Λ (197) and the spectral density correction $\Delta\rho$ (204). For this purpose we approximate the factor $\cosh^{-2} z$ by a rectangle of width $2b$ and high a which is also symmetric around $z = 0$. The area under $\cosh^{-2} z$ is given as $\int dz \cosh^{-2} z = 2$; nevertheless we shall leave the area of the rectangle unspecified for now. Therefore (197) changes to

$$\cosh^{-2} z \to a\Theta(b - |z|) \tag{226}$$

$$\Lambda_K = \Lambda - a\Theta(b - |z|)\frac{\delta(\Lambda)}{2} \tag{227}$$

Here we must require that $\frac{a}{2} \leq 1$ to ensure that (197) is still a contraction for finite Λ and the Banach fixed point theorem is still applicable. With this the correction (204) changes to

$$\Delta\rho(k) = -\Theta_{(\Lambda-\frac{a}{2}\delta,\Lambda)}(|k|)\int_{-b}^{b}\frac{dz}{2\pi}\frac{-a\Theta(b-|z|)+k^2+1}{k^2+1} \tag{228}$$

$$= -\Theta_{(\Lambda-\frac{a}{2}\delta,\Lambda)}(|k|)\frac{1}{\pi}[b - \frac{ab}{k^2+1}] \tag{229}$$

The integral of the spectral density correction is now given by

$$\int dk\Delta\rho = -\frac{1}{\pi}2\int_{\Lambda-\frac{a}{2}\delta}^{\Lambda} dk[b - \frac{ab}{k^2+1}] \tag{230}$$

$$= -\frac{1}{\pi}ab[\delta(\Lambda) - 2[\arctan\Lambda - \arctan(\Lambda - a\frac{\delta}{2})] \tag{231}$$

$$= -\frac{1}{\pi}[ab\delta(\Lambda) - 2ab\frac{a}{\Lambda^3} + O(\frac{a^2}{\Lambda^5})] \tag{232}$$

In the last step we have expanded the second term around $\Lambda = \infty$. The first term in the bracket corresponds to (213) and the second term to (215). This coincides with the undeformed result (213) if and only if $ab = 1$ and thus when the area of the rectangle $2ab = 2$ equals the area under the function $\cosh^{-2} z$. Analogously, the correction to the quantum mass of the kink does not change if $ab = 1$. The analogue of (218) using (229) reads:

$$\Delta M = \frac{\hbar m}{2}\int dk\sqrt{k^2+1}\Delta\rho(k) = -\frac{\hbar m}{\pi}\int_{\Lambda-\frac{a}{2}\delta}^{\Lambda} dk\left(b\sqrt{k^2+1} - \frac{ab}{\sqrt{k^2+1}}\right) \tag{233}$$

$$= -\frac{\hbar m}{\pi}(\frac{b}{2}[\Lambda\sqrt{\Lambda^2+1} - \Lambda\sqrt{(\Lambda - a\frac{\delta}{2})^2+1}] + ab\frac{\delta(\Lambda)}{4}\sqrt{(\Lambda - a\frac{\delta}{2})^2+1} \tag{234}$$

$$+\frac{b}{2}[\mathrm{arcsinh}\Lambda - \mathrm{arcsinh}(\Lambda - a\frac{\delta}{2})] - ab[\mathrm{arcsinh}\Lambda - \mathrm{arcsinh}(\Lambda - a\frac{\delta}{2})]) \tag{235}$$

$$\stackrel{\Lambda\to\infty}{\longrightarrow} -\frac{\hbar m}{\pi}ab(\frac{1}{2}+\frac{1}{2}+0+0) = -\frac{\hbar m}{\pi}ab \tag{236}$$

This gives again the correct result (219) if and only if the area of the rectangle equals the area of $\cosh^{-2} z$, i.e. if $ab = 1$. It seems to be arbitrary where to locate spatially the modification of the kink spectrum as long as the average is the same. Therefore the equality between the two projectors (196) does not have to be a strong relation, i.e. a identity between operators,

but rather a weak relation, i.e. an identity between their action on states, that is, a relation between distributions.
All the above considerations work in full analogy for the ϕ^4 model and give also the correct results.

Discussion

In conclusion we can say that we have a working principle that relates the regularization parameter of the nontrivial Λ_K to that of the trivial sector Λ in a way that both sectors are regularized in a consistent way, i.e. so that one can really compare them with each other. This is the main point in our opinion: to find such a relation so that one can regularize the nontrivial sector consistent with the vacuum sector, in which also the renormalization is fixed and defines the physical parameters (mass, coupling,..) of the theory. Again the big advantage of our principle is, besides that it gives the correct results in a consistent way, that it is not restricted to two dimensions or supersymmetric theories. It is to expect that it also works for fermions. This work is in progress.

Nevertheless further investigations are in order. Especially the identification of the regularized units has to be investigated from a mathematical point of view. For the "off diagonal" term these are very complicated integral equations, similar to Fredholm equations, for the unknown function $\Lambda_K(\Lambda)$. It would be worth to investigate this further to find out under what conditions a solution for $\Lambda_K(\Lambda)$ exists. It is conceivable that in general a strict cutoff-function (step function) as used in our ansatz does not solve this problem.

5 Fermions

Until now we have only considered bosonic fields and treated them in the path integral as classical functions. The nature (physics) of fermionic fields is completely different. While classical Bose fields are found in nature (e.g. electromagnetic waves, gravitational fields, etc.) classical Fermi fields are not, at least not in the same sense. From the quantum-point of view a collection of a very large number of bosons in more or less the same quantum state, i.e. a coherent state, can be described by a classical (on-shell) field and also observed as such fields, if one does not look too closely. The same cannot happen for fermionic fields since the *Pauli exclusion principle* forbids more than one fermion per state. This "strange" behavior is respected in description of "classical" (not operators) fermionic fields by a "strange" algebra of the fermionic degrees of freedom, namely Grassmann algebras. Heuristically this can be obtained by the formal limit $\hbar \to 0$ in the anticommutation relations of Dirac fields:

$$\{\psi_\alpha(x,t), \psi^\dagger_\beta(x',t)\} = \hbar\delta_{\alpha\beta}\delta(x-x')$$
$$\{\psi_\alpha(x,t), \psi_\beta(x',t)\} = \{\psi^\dagger_\alpha(x,t), \psi^\dagger_\beta(x',t)\} = 0.$$

By this point of view classical fermionic fields are functions over the space-time, parametrized by (x,t) which have their values in a Grassmann algebra. This limit does not describe the physical world in an approximative sense as mentioned above. To describe the (quantum) dynamics of a system we not only need the functional dependence on the degrees of freedom (DOF) on the the space-time parameters but we also need the dependence of dynamical

quantities like the action or the Hamiltonian on the fields (coordinates for finite systems). Since fermionic DOF are elements of a Grassmann algebra we need a generalization of operations like variation and path-integration to "functions" on a Grassmann algebra. For this we consider some basic properties.

5.1 Grassmann calculus

5.1.1 Grassmann algebras.

A finite dimensional Grassmann algebra $\mathcal{G}_N(\mathbb{K})$ over the field [35] $\mathbb{K}$can be constructed from a set of N elements $\{a_1, \ldots, a_N\}$, called *generators* which fulfill the following algebra:

$$\{a_i, a_j\} = 0 \; \forall i, j = 1, \ldots, N.$$

This relation is invariant under general linear transformations $a_i \to G_{ij} a_j$, where the matrix entries $G_{ij} \in \mathbb{K}$. The whole algebra is a 2^N dimensional vector space in which the ordered products

$$\begin{aligned}
&1\\
&\{a_i | i = 1 \ldots N\}\\
&\{a_i a_j | i < j; i, j = 1 \ldots N\}\\
&\{a_i a_j a_k | i < j < k; i, j, k = 1 \ldots N\}\\
&\vdots\\
&a_1 a_2 \ldots a_N,
\end{aligned}$$

form a basis. Concrete realizations of this algebra are for example the exterior algebra of forms over a N-dimensional vector space or the algebra of N fermionic excitation operators acting on a Fock space. The algebra is the direct sum of an even and an odd part,

$$\mathcal{G} = \mathcal{G}_+ \oplus \mathcal{G}_-,$$

and thus a Grassmann algebra is a $\mathbb{Z}_2$ graduated algebra. The even part consists of all linear combinations of basis-elements which consists of an even number of generators and analogously the odd part is the linear hull of basis-elements consisting of an odd number of generators. Both, $\mathcal{G}_\pm$, have special features.

The even part $\mathcal{G}_+$ is a commutative sub algebra, i.e.

$$\begin{aligned}
\text{for } f_+, g_+ \in \mathcal{G}_+ \quad \Rightarrow \qquad & f_+ g_+ \in \mathcal{G}_+\\
& [f_+, g_+] = 0.
\end{aligned}$$

Therefore one can define functions on $\mathcal{G}_+$ and multiply and add them in the usual way. For example an element of "degree" two

$$W = \sum_{i,j=1, i<j}^{N} \Delta_{ij} a_i a_j,$$

[35] $\mathbb{K}$ will be mostly equal to $\mathbb{C}$

where the coefficients $\Delta_{ij} \in \mathbb{K}$ are ordinary numbers and antisymmetric in the indices i, j, one can define an exponential function, e.g.

$$e^{iW} = \sum_{m \geq 0} \frac{(i)^m}{m} W^m \in \mathcal{G}_+, \tag{237}$$

where $W^0 := 1$ per definition. For a finite dimensional Grassmann algebra this series truncates at $2m \geq N$. Also usual function-equations like

$$e^S e^T = e^{S+T} \quad \text{for } S, T \in \mathcal{G}_+ \tag{238}$$

are meaningful. Only the existence of an inverse element is not guaranteed in general. But for the exponential (237) even the inverse exists, i.e.

$$e^{iW}\ e^{-iW} = 1.$$

The dynamical quantities like the action (Lagrangian) will be such even functions, so that we can work with them in the usual way.

The odd part $\mathcal{G}_-$ where as is not closed under multiplication (the product of two odd numbers can be even) but the elements of $\mathcal{G}_-$ are nilpotent, i.e

$$\psi \in \mathcal{G}_- \quad \Rightarrow \quad (\psi)^2 = \psi\psi = 0$$

The above considerations are also valid for the infinite dimensional case $N \to \infty$ and this is the case of interest for quantized fermionic degrees of freedom. For pure "classical" (on shell) considerations one can describe a system of N fermionic DOF within the framework of *Grassmann mechanics* (pseudo-classical mechanics) as elements of an N-dimensional Grassmann algebra ([17],[15]). This is not possible for quantum considerations within the path integral, where the fermionic DOF do not become operators, even for finite degrees of freedom (not field theory) as we will see. To describe the evolution of a system we are of course interested in Grassmann valued *functions*, i.e. objects of the form

$$\begin{aligned} f: \quad & \mathcal{B} \to G_\infty \\ & x \to f(x), \end{aligned}$$

where $\mathcal{B}$ is the parameter domain. With regard to path integral quantization we have already chosen an infinite dimensional Grassmann algebra. For a N-dimensional (pseudo) mechanical system thus one has to consider functions of the form

$$q^i(t) = \sum_k f_k^i(t) a_k,$$

where $\{f_k^i\}$ form a complete set in a infinite dimensional function space. The use of the sum is a priori a symbolic notation, but will coincide with concrete expressions due to regularization. Analogously one obtains for fermionic fields

$$\psi(x,t) = \sum_k \psi_k(x,t) a_k,$$

where again the the functions $\{\psi_k(x,t)\}$ form a complete set in an infinite dimensional function space. The index set $\{k\}$ in the case of fields is of course larger than for finite DOF. Of

particular interest are spinor fields $\psi_\alpha(x,t)$ on a D-dimensional Minkowski space, so that the underlying function space is the set of square integrable functions $\psi : \mathcal{M}_D \to \mathbb{C}^{2^{[D/2]}}$ in which the components $\{\psi_{\alpha,k}(x,t)\}$ form a complete set. The hermitian adjoint field writes as

$$\psi^\dagger(x,t) = \sum_k \psi_k^\dagger(x,t)\bar{a}_k,$$

where the $\{\bar{a}_k\}$are independent of the $\{a_k\}$, so that the algebra $\mathcal{G}_\infty$ is generated by the infinite set of generators $\{a_k, \bar{a}_k | k = 1, \dots\}$. By this the fields ψ and $\psi^\dagger$ are treated as independent degrees of freedom. Usually one decomposes the spinors ψ_k into positive and negative frequency parts as well as according to different spin. This is absorbed in the "master" index k. When one studies the Dirac equation in the sense of "first quantization", i.e. a one particle wave equation, one solves the Dirac equation for the components $\psi_k(x,t)$. These solution, as for example for hydrogen-like atoms, are in the sense of path integral quantization "classical" solutions of the system. Thus "second" quantization is the path integration of quantum fluctuations around these classical solutions.

5.1.2 Variation and integration

The Lagrangian of a theory is a composite object of the degrees of freedom. In the bosonic case thus it can be treated as a normal function depending on the fields or coordinates. To adopt Lagrangian methods to Grassmann valued fields we have to consider functions defined on the Grassmann algebra, i.e.

$$\begin{aligned} L: \quad & D(\mathcal{G}) \to I(\mathcal{G}) \\ & g \to L(g), \end{aligned}$$

where $D(\mathcal{G}), I(\mathcal{G})$ are the domain and the image, respectively. In general each analytic function can be generalized to a *super-analytic* one [12]. For a general Grassmann number of degree N, which includes L and g, which are of the form (to make this expansion unique the coefficients have to be antisymmetric)

$$f = f_0 + f_i a_i + \frac{1}{2!} f_{ij} a_i a_j + \dots + \frac{1}{p!} f_{i_1,\dots,i_p} a_{i_1} \dots a_{i_p}, \tag{239}$$

one can define a norm $||\ ||$ as follows [12]:

$$||f|| = |f_0| + \sum_{p=1}^{N} \sum_{i_1 < \dots i_p} |f_{i_1 \dots i_p}|^2.$$

Thus one has a topology on $\mathcal{G}$ and therefore the concept of "being close to". In the following the Lagrangian L will exclusively be bilinear in fermionic DOF. Thus we restrict ourselves to the consideration of Lagrangians of the form

$$L(f_\alpha, \dot{f}_\alpha, g_\alpha, \dot{g}_\alpha) = f_\alpha D_{\alpha\beta} g_\beta,$$

where $D_{\alpha\beta}$ is a matrix valued differential operator and f_α, g_α are tuples (e.g. spinors) of Grassmann valued functions of the form

$$f_\alpha = \sum_i (f_\alpha)_i a_i \quad , \quad g_\alpha = \sum_i (g_\alpha)_i b_i. \tag{240}$$

Thus the Grassmann algebra is generated by the set $\{a_i, b_i\}$ where a priori all a_i, b_i are different. Thus under a variation $\delta f_\alpha = \sum_i \delta(f_\alpha)_i a_i$ the Lagrangian changes by

$$\delta L = \delta f_\alpha D_{\alpha\beta} g_\beta = -D_{\alpha\beta} g_\beta \delta f_\alpha,$$

and analogously for a variation of g_α, where derivatives must be partially integrated in the action as usual. Thus the variational calculus is very similar to bosonic DOF, the only thing one has to care for is the ordering of the DOF. One can also define differentiations w.r.t. Grassmann variables, which act as derivations on the algebra, but these are rather formal operations, i.e. they are defined purely algebraically (see for example [16],[18]). The reason for this is that it is not possible to define differentiation as a limit of differential quotients, since the inverse of a Grassmann number, especially for Grassmann numbers like (240), is in general not defined (the inverse exists only if the *body* f_0 is unequal zero [12]).

One can also define a formal integration on the Grassmann algebra, which like differentiation is purely algebraic. The so called *Berezin* integral is defined by the following axioms ([16], [18]): On a N- dimensional Grassmann algebra $\mathcal{G}_\mathcal{N}$, generated by $\{a_i\}$, the *linear* functionals $\int da_i : \mathcal{G}_+ \to \mathbb{K}$ are defined as follows

1. $\int da_i(1) = 0$
2. $\int da_i(a_i) = 1$
3. $\{da_i, a_j\} = 0$ for $i \neq j$
4. $\{da_i, da_j\} = 0$ for$i \neq j$

With this definitions one has for example

$$\int da_1 \cdots \int da_N f := \int da_1 \dots da_N f = f_{1,\dots,N},$$

where $f_{1,\dots,N}$ is the "highest" component in the expansion analogous to (239) of f. Of particular interest are integrations of exponential functions of the form

$$\int da_1 db_1 \dots da_N db_N \; e^{-\sum \lambda_k a_k b_k} = \int da_1 db_1 \dots da_N db_N \prod_{k=1}^{N} e^{-\lambda_k a_k b_k} \tag{241}$$

$$= \prod_{k=1}^{N} \int da_k db_k (1 - \lambda_k a_k b_k) \tag{242}$$

$$= \prod_{k=1}^{N} \lambda_k \int da_k a_k db_k b_k = \prod_{k=1}^{N} \lambda_k. \tag{243}$$

In the first two lines we have used (238) and (237). In the last line we have applied the rules $1, 3$ and 2.

5.2 The Grassmann oscillator, fermionic boundary conditions

We consider a hermitian fermionic oscillator. Its Lagrangian is given by

$$L = \frac{i}{2}\left(a^\dagger \dot{a} - \dot{a}^\dagger a\right) - \omega a^\dagger a \tag{244}$$

$$= \frac{1}{2} a^\dagger (i\partial_t - \omega) a + \frac{1}{2} a (i\partial_t + \omega) a^\dagger. \tag{245}$$

The fermionic DOF $a^\dagger, a$ are elements of the infinite dimensional Grassmann algebra $\mathcal{G}_\infty$ which is generated by the infinite set $\{a_k, \bar{a}_k\}$. Thus $a^\dagger(t), a(t)$ are of the form

$$a(t) = \sum_k f_k(t) a_k \quad , \quad a^\dagger(t) = \sum_k f_k^*(t) \bar{a}_k. \tag{246}$$

5.2.1 Variation principle

There exists a fundamental difference between fermionic and and bosonic DOF. Besides being anticommuting fermionic DOF are first order systems, i.e. the e.o.m are first order differential equations. In the Lagrangian this is reflected in the fact that the velocities occur linearly. In the canonical formalism this leads to so called constraints. For the variation principle this results in the need of introducing surface terms for the action to be able to define a consistent variation principle [21]. Variation principles fit perfectly to the principles of quantum theory, since they fix initial and final positions (vanishing variation) rather than position and velocity at the same time. Thus they lead to boundary value problems rather than to initial value problems. But for first order systems this is problematic, since fixing the initial and final values overconstrains a first order differential equation. For field systems this generalizes to spatial boundaries. Therefore one needs a modified variation principle which is consistent with first order systems and leads to the classical e.o.m. This is done by fixing a linear combination of the boundary values rather than each of them separately, for example

$$\begin{aligned} a(t') + \sigma a(t'') = \xi = const \quad &\Rightarrow \quad \delta[a(t') + \sigma a(t'')] = 0 \\ a^\dagger(t') + \sigma a^\dagger(t'') = \xi^\dagger = const \quad &\Rightarrow \quad \delta[a^\dagger(t') + \sigma a^\dagger(t'')] = 0, \end{aligned}$$

where $\xi, \xi^\dagger$ are constant Grassmann numbers. But for this variation principle to give the correct e.o.m. one has to introduce surface terms in the action. For the action

$$S = \int_{t'}^{t''} dt L + \sigma \frac{i}{2} [a(t') a^\dagger(t'') + a^\dagger(t') a(t'')]$$

this variation principle leads to following e.o.m.:

$$\begin{aligned} \delta a^\dagger: \quad & i\dot{a} - \omega a = 0 \quad && \text{with BC } a(t') + \sigma a(t'') = \xi \\ \delta a: \quad & i\dot{a}^\dagger + \omega a^\dagger = 0 \quad && \text{with BC } a^\dagger(t') + \sigma a^\dagger(t'') = \xi^\dagger. \end{aligned}$$

As one can see for the classical "paths" $a, a^\dagger$ only the boundary term contributes to the action:

$$S_{cl} = \frac{i}{2} [a(t') \xi^\dagger + a^\dagger(t') \xi]. \tag{247}$$

For Dirac fields this generalizes as follows: The variation principle is defined as

$$\begin{aligned} \psi|_{B''_\mu} - \Gamma_{(\mu)} \psi|_{B'_\mu} = \xi_{B_\mu} \quad &\Rightarrow \quad \delta \left[\psi|_{B''_\mu} - \Gamma_{(\mu)} \psi|_{B'_\mu} \right] = 0 \\ \psi^\dagger|_{B''_\mu} - \Gamma_{(\mu)} \psi^\dagger|_{B'_\mu} = \xi^\dagger_{B_\mu} \quad &\Rightarrow \quad \delta \left[\psi^\dagger|_{B''_\mu} - \Gamma_{(\mu)} \psi^\dagger|_{B'_\mu} \right] = 0, \end{aligned}$$

where B'_μ, B''_μ are the boundaries, spatial or of the time interval, and $\xi_{B_\mu}, \xi^\dagger_{B_\mu}$ are constant Grassmann spinors, for the particular boundaries of the "direction" B_μ. $\Gamma_{(\mu)}$ are constant matrices[36]. It can be shown that for classical fields, i.e. those fulfilling the classical e.o.m., again

[36] Depending on the explicit form of the Lagrangian the matrices $\Gamma_{(\mu)}$ have to fulfill certaint relations, so that the above variation principle give the e.o.m.

only boundary terms contribute to the action and this contribution are again proportional to the constant spinors, i.e.

$$S_{cl} \sim \xi_{B_\mu}, \xi^\dagger_{B_\mu}.$$

Thus the fields fulfilling "homogeneous" boundary conditions, i.e. $\xi = 0 = \xi^\dagger$ give no boundary contributions to the action., even if they do not fulfill the e.o.m. For unbounded space-time regions one does not have to care about these things, since because of natural boundary conditions all contributions vanish at infinity.

5.2.2 Spectral function and energy spectrum

As can be shown in the homomorphic representation of the Grassmann algebra of $a, a^\dagger$, the trace of the time evolution operator and thus the spectral function is given by the path integral of antiperiodic paths [22]. Thus the boundary contributions to the action are zero and one obtains for the oscillator

$$\mathrm{Tr}\left[e^{-\frac{i}{\hbar}HT}\right] = K(T) \int\limits_{antiperiodic} \mathcal{D}a^\dagger \mathcal{D}a \; e^{\frac{i}{\hbar}\int_T dtL}. \tag{248}$$

The Lagrangian L is given by (245) and $K(T)$ is an appropriate measure constant. To perform the path integration we have to diagonalize the action. For this we determine the coefficient functions in (246) so that they solve the eigenvalue problem[37]

$$(i\partial_t - \omega) f_k = \epsilon_k f_k \quad , \quad f_k(t+T) = -f_k(t).$$

The solution is easily obtained and given by

$$f_k(t) = \frac{1}{\sqrt{T}} e^{ip_k t} \quad \text{with} \quad p_k = \frac{(2k+1)\pi}{T}, \tag{249}$$

$$\epsilon_k = -(p_k + \omega) \quad , \quad k = 0, \pm 1, \pm 2, \ldots \tag{250}$$

The complex conjugate coefficient function in (246) automatically fulfills

$$(i\partial_t + \omega) f_k^* = -\epsilon_k f_k^*,$$

where ϵ_k is given by (250). Inserting this into the Lagrangian (245) the action can be written as

$$S = \int_T dtL = \sum_k \bar{a}_k a_k \epsilon_k.$$

For regularization k takes only a finite number of values, i.e. $k = -N, \ldots, N$. The measure is defined as

$$\int \mathcal{D}a^\dagger \mathcal{D}a := K(T) \prod_{-N}^{N} \int d\bar{a}_k da_k,$$

[37] Since the classical solutions do not contribute to the action, this is equivalent to an expansion arround around these classical solutions.

where $K(T)$ will be defined by a suitable normalization condition. With (243) one obtains for the trace (248)

$$K(T)\prod_{-N}^{N}\int d\bar{a}_k da_k \frac{i}{\hbar}\bar{a}_k a_k \epsilon_k \tag{251}$$

$$= \left[K(T)\prod_{-N}^{N}\frac{i}{\hbar}\frac{T}{(2k+1)\pi}\right]\prod_{-N}^{N}\left(1+\frac{\omega T}{(2k+1)\pi}\right) \tag{252}$$

$$\underset{N\to\infty}{\longrightarrow}\lim_{N\to\infty}\left[K(T)\prod_{-N}^{N}\frac{-i}{\hbar}\frac{T}{(2k+1)\pi}\right]\cos\frac{\omega T}{2}. \tag{253}$$

The closed formula for the second product in (252) is given in [25]. As in the bosonic case, the measure constant $K(T)$ is purely kinetic and does not exist by itself. We normalize it as follows

$$\lim_{N\to\infty}\left[K(T)\prod_{-N}^{N}\frac{i}{\hbar}\frac{T}{(2k+1)\pi}\right]=2.$$

Thus we get for the trace

$$\mathrm{Tr}\left[e^{-\frac{i}{\hbar}HT}\right]=\sum_n e^{-\frac{i}{\hbar}E_n}=e^{i\frac{\omega T}{2}}+e^{-i\frac{\omega T}{2}}. \tag{254}$$

The spectrum of the fermionic oscillator consists only of two levels. From (254) one obtains the ground state (lowest level)

$$E_0=-\frac{\hbar\omega}{2},$$

and it is same as for an bosonic oscillator but with opposite sign.

5.3 Mode regularization including fermions

Now we consider the supersymmetric extension of the SG and ϕ^4 model, respectively. As mentioned in the introduction we will not stress the supersymmetry of the system (up to some fundamental properties) and mainly concentrate on the influence of the nontrivial background on fermions in the regularization/renormalization procedure.

5.3.1 Classical properties

The supersymmetric extension of Lagrangians of the form

$$\mathcal{L}=\frac{1}{2}[\partial\phi)^2-V^2(\phi)] \tag{255}$$

is given by

$$\mathcal{L}=\frac{1}{2}[(\partial\phi)^2-V^2(\phi)]+\frac{1}{2}\bar{\psi}[i\partial\!\!\!/-V'(\phi)]\psi$$

where ψ is a Majorana spinor field and $V(\phi)$ is related to the original potential $U(\phi)$ via $V = 2\sqrt{U}$. For SG and ϕ^4 it is given by

$$V_\phi(\phi) = \sqrt{\frac{\lambda}{2}}(\phi^2 - \frac{\mu^2}{\lambda}) \tag{256}$$

$$V_{SG}(\phi) = \frac{2\mu}{\sqrt{\gamma}} \sin \frac{\sqrt{\gamma}\phi}{2} \tag{257}$$

The associated action is invariant under the (rigid) SUSY transformations

$$\begin{aligned} \phi \to \phi + \delta\phi: \quad & \delta\phi = \bar{\epsilon}\psi \\ \psi \to \psi + \delta\psi: \quad & \delta\psi = [i\partial\!\!\!/ - V(\phi)]\epsilon \end{aligned}$$

where ϵ is a constant Grassmann spinor. The classical equations of motion are

$$\begin{aligned} \Box\phi + V(\phi)V'(\phi) + \frac{1}{2}\bar{\psi}\psi V''(\phi) = 0 \\ [i\partial\!\!\!/ - V'(\phi)]\,\psi = 0 \end{aligned}$$

and there are the following classical (kink) solutions

$$\text{fermion vacuum}: \quad \phi = \phi_{K_\sigma} \quad \psi = 0 \tag{258}$$

$$\text{fermionic zero} - \text{mode}: \quad \phi = \phi_{K_\sigma} \quad \psi^\sigma = -\sigma\phi'_{K_\sigma} P_\sigma \epsilon \tag{259}$$

where ϕ_{K_σ} are the (anti)kinks (148). The second solution can be obtained by a SUSY transformation of the first one and was first given by [44],[45]. The projector $P_\sigma = \frac{1}{2}[\mathbb{1} - \sigma i\gamma^1]$ acts on the constant Grassmann spinor ϵ. From this one can see that the ground state $\{\phi_{K_\sigma}, \psi = 0\}$ is invariant under the half SUSY transformation with parameters $P_\sigma\epsilon = 0$.

For the following calculations we choose a Majorana representation $\gamma^0 = \sigma_2 \;\; \gamma^1 = i\sigma_1$ of the Clifford algebra. With this choice we have $\gamma_\star := \gamma^0\gamma^1 = -\sigma_1$. The intertwiners for spinors are

$$\bar{\psi} = \psi^\dagger\gamma^0 \qquad \psi^c = \psi^*$$

The Majorana condition therefore simply becomes $\psi^* = \psi$.

5.3.2 Vacuum sector and Renormalization

The trivial (vacuum) solutions are given as

$$SG: \quad \psi_V = 0 \quad \phi_V = 0 \tag{260}$$

$$\phi^4: \quad \psi_V = 0 \quad \phi_V = \frac{m}{\sqrt{2\lambda}} \tag{261}$$

Expanding the action around these solutions leads to standard perturbation theory (Feynman graphs) and one obtains in a minimal renormalization scheme ($m^2 = m_0^2 - (\delta m^2)_{susy}$) the following counter-terms [37]

$$SG: \qquad (\delta m^2)_{susy} = \hbar\frac{\gamma m^2}{8\pi}\int_0^\Lambda \frac{dk}{\sqrt{k^2+m^2}}$$

$$\phi^4: \qquad (\delta m^2)_{susy} = \hbar\frac{\lambda}{2\pi}\int_0^\Lambda \frac{dk}{\sqrt{k^2+m^2}}$$

The renormalization conditions are that the bosonic seagull loop for the fermionic two-point function vanishes (SG) and the bosonic and fermionic tadpole do not contribute (ϕ^4), respectively. Because of the change of δm^2 also the counter-term contribution in the kink sector (156) is different in the SUSY case. In both cases one has $\delta M_{susy} = \hbar m \int_0^\Lambda \frac{dz}{2\pi} \frac{1}{\sqrt{z^2+1}} := \delta M_B + \delta M_F$

$$SG: \quad \delta M_F = -\hbar m \int_0^\Lambda \frac{dz}{2\pi} \frac{1}{\sqrt{z^2+1}} \tag{262}$$

$$\phi^4: \quad \delta M_F = -2\hbar m \int_0^\Lambda \frac{dz}{2\pi} \frac{1}{\sqrt{z^2+1}} \tag{263}$$

The quadratic part of the expanded fermionic Lagrangian is given as

$$\mathcal{L}_\psi = \frac{1}{2}\bar{\psi}[i\partial\!\!\!/ - V'(\phi_V)]\psi + O(\bar{\psi}\psi\eta) \tag{264}$$

In both models one has $V'(\phi_V) = m$ so that for SG and ϕ^4 the quadratic fermionic Lagrangian is given by

$$\mathcal{L}_\psi^{(2)} = \frac{1}{2}\bar{\psi}[i\partial\!\!\!/ - m]\psi \tag{265}$$

Boundary conditions

Applying our symmetry principle on (261) we get with the ansatz ($A = \pm 1$)

$$\psi(-L/2) = A\psi(L/2) \tag{266}$$

for the change of the quadratic Lagrangian when transported around the compactified dimension

$$x \to x - L \Rightarrow \delta\mathcal{L} = 0$$

Therefore we have to use P/AP BC in the vacuum sector so that the action gets no boundary contribution. By contrast, twisted (anti)periodic BC in the vacuum sector as used in [31], $\psi(-L/2) = A\gamma_\star\psi(L/2)$, induce a boundary contribution to the quadratic action $S_\psi^{(2)} = \int_{T\mathrm{x}L} dtdx\mathcal{L}_\psi^{(2)}$ of the form $\sim \int_L dx\delta\mathcal{L} = \int_L dx 2m\bar{\psi}\psi$ and therefore also to the energy.

All the following results do not depend on the choice of our symmetry BC (266) as it should be for BC that do not induce boundary contributions. This is analogous to the bosonic case.

5.3.3 Spectral function

In the action expanded around the vacuum (261), $S^{(2)}[\eta, \psi, \bar{\psi}]$, no interaction between the bosonic and fermionic fluctuations occurs. So we can calculate the fermionic contribution to the spectral function separately:

$$Tr_V e^{-\frac{i}{\hbar}HT}|_\psi = \int_{antiperiodic} \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{\frac{i}{\hbar}\int_{LxT} dtdx \frac{1}{2}\bar{\psi}[i\partial\!\!\!/ - m]\psi} + O(\hbar^2) \tag{267}$$

Here one has to integrate over fields which are antiperiodic in time [22]. We diagonalize the spatial part of the operator in the action $\psi^\dagger[i\partial_t + i\gamma_\star\partial_x - \gamma^0 m]\psi$, which gives (with our representation of the γ's) the following spectrum

$$\psi_\pm(k,x) = \frac{1}{\sqrt{2L}} \begin{bmatrix} 1 \\ \pm\frac{1}{\omega}(k - im) \end{bmatrix} e^{ikx} \quad \omega_\pm = \pm\omega = \pm\sqrt{k^2+m^2} \tag{268}$$

The BC quantize the momenta as

$$\psi_\pm(k, x+L) = A\psi_\pm(k,x) \quad \rightarrow \quad Lk_n^A = (2n+A)\pi \tag{269}$$

so that a finite (symmetric) expansion gives for the fermionic field

$$\psi(x,t) = \sum_{-(N-A)}^{N} (a_n(t)\psi_+(k_n,x) + b_n(t)\psi_-(k_n,x)) \tag{270}$$

which now automatically fulfills the BC (266), i.e. $\psi(x+L,t) = A\psi(x,t)$. The time dependent coefficients in (270) are Grassmann-valued functions.

Majorana condition

Now we have to set up the Majorana condition $\psi^*(x,t) = \psi(x,t)$. From (268) and (269) we see that

$$\begin{aligned} \psi_\pm^*(k_n,x) &= \psi_\mp(-k_n,x) \\ -k_n^A &= k^A_{-(n+A)} \end{aligned}$$

So the Majorana condition for (270) gives

$$\begin{aligned} a_n^*(t) &\stackrel{!}{=} b_{-(n+A)}(t) \\ b_n^*(t) &\stackrel{!}{=} a_{-(n+A)}(t) \end{aligned}$$

These two conditions are compatible and therefore the Majorana condition for the field is fulfilled for all times

$$\psi(x,t) = \sum_{-(N-A)}^{N} \left(a_n(t)\psi_+(k_n^A,x) + a_n^*(t)\psi_+^*(k_n,x)\right)$$

Inserting this field in the action in (267) one obtains

$$S_\psi^{(2)} = \sum_{-(N-A)}^{N} \int_T dt \left[\frac{i}{2}(a_n^*\dot{a}_n + a_n\dot{a}_n^*) + \omega(k_n^A)a_n^*a_n\right]$$

This is the sum of $2N+1+A$ Grassmann-oscillators with the frequencies $\omega_F^V = \omega(k_n^A)$. Note that a complex conjugated pair forms one degree of freedom.

Ground-state energy

The measure in (267) is

$$\mathcal{D}\bar{\psi}\mathcal{D}\psi = (K(T))^n \Pi da_n^* da_n$$

and as in the bosonic case independent of the considered topological sector. And also as in the bosonic case there exists a subtlety due to zero modes in the nontrivial sector. But now it weighs much more as in the bosonic case (see below). Performing the path integration one can read off the ground state energy of the vacuum:

$$\text{ground}-\text{state energy}: \qquad E_V = -\frac{\hbar m}{2} \sum_{-(N-A)}^{N} \sqrt{\left(\frac{(2n+A)\pi}{Lm}\right)^2 + 1} \tag{271}$$

$$\text{mode number}: \qquad \#_V = 2N + 1 + A \tag{272}$$

$$\text{energy cutoff}: \qquad \Lambda_A = k_N^A = \frac{(2N+A)\pi}{Lm} \tag{273}$$

So one has up to the sign the same ground state energy as in the bosonic vacuum sector, as it would be expected by supersymmetry.

5.3.4 Kink sector

The treatment of the kink sector is analogous to the vacuum sector, but more involved and with some additional subtleties. For the semi-classical calculation (one loop) we expand the Lagrangian around the stable kink ground state[38]

$$\{\phi = \phi_K, \psi = 0\}$$

The influence of the kink in the quadratic Lagrangian (264) is thus given by $V'(\phi_K)$ which reads in our dimensionless variables $z = \frac{mx}{l}$ for both models ($l = 1, 2$ for SG, ϕ^4)

$$V_l'(\phi_K^l) = m \tanh z \tag{274}$$

Therefore the fermionic quadratic action is given by

$$S_\psi^{(2)} = \frac{1}{2} \int_T dt \frac{l}{m} \int_{\tilde{L}} dz \psi^\dagger(z,t)[i\partial_t + i\gamma_\star \partial_z - \gamma^0 m \tanh z]\psi(z,t) \tag{275}$$

To perform the path integration (267), but now for the kink sector, we diagonalize the spatial part of (275), where we normalize the eigen-states properly, so that the factor $\frac{l}{m}$ in (275) is canceled. The eigen value-problem to be solved is thus

$$\begin{bmatrix} 0 & i\frac{m}{l}A_l^\dagger \\ -i\frac{m}{l}A_l & 0 \end{bmatrix} \begin{bmatrix} \xi \\ \rho \end{bmatrix} = \omega \begin{bmatrix} \xi \\ \rho \end{bmatrix} \tag{276}$$

with the operators

$$A_l = \partial_z + l \tanh z$$
$$A_l^\dagger = -\partial_z + l \tanh z$$

[38] If one expands the action around the other classical configuration where $\psi = \psi_{zero} \neq 0$ the fermionic and bosonic fluctuations interact already in the quadratic action.

The operator in (276) is self adjoint w.r.t. the scalar product

$$< \chi|\psi >= \int_{\tilde{L}} dz \chi^\dagger \psi = \int_{\tilde{L}} dz (\chi_1^* \psi_1 + \chi_2^* \psi_2)$$

if the surface term $(\chi_1^*\psi_1+\chi_2^*\psi_2)_{\tilde{L}/2}-(\chi_1^*\psi_1+\chi_2^*\psi_2)_{-\tilde{L}/2}$ vanishes. This is true for the following spin structures (anti/periodic and twisted anti/periodic, $A = \pm 1$)

$$\begin{aligned}
(P/AP): \quad & \psi(-L/2) = A\psi(L/2) \rightarrow \quad \begin{bmatrix} \xi \\ \rho \end{bmatrix} (-\frac{\tilde{L}}{2}) = A \begin{bmatrix} \xi \\ \rho \end{bmatrix} (\frac{\tilde{L}}{2}) \\
(TP/TAP): \quad & \psi(-L/2) = A\gamma_\star \psi(L/2) \rightarrow \quad \begin{bmatrix} \xi \\ \rho \end{bmatrix} (-\frac{\tilde{L}}{2}) = -A \begin{bmatrix} \rho \\ \xi \end{bmatrix} (\frac{\tilde{L}}{2})
\end{aligned}$$

The extra minus for (TP/TAP) is due to our metric signature $(+,-)$. The coupled system of differential equations (276) can be decoupled by expressing the lower component ρ through the upper component ξ. So one obtains

$$A_l^\dagger A_l \xi = \frac{l^2}{m^2}\omega^2 \xi =: E_{(l)}\xi \rightarrow \text{ eigen values } \omega_\pm =: \pm\omega = \pm\frac{m}{l}\sqrt{E_{(l)}} \tag{277}$$

$$\rho_\pm =: \pm\rho = \pm(\frac{-i}{\sqrt{E_{(l)}}} A_l \xi) \text{ iff } E \neq 0 \tag{278}$$

The differential equation for ξ is the same as for the bosonic fluctuations and the lower component ρ is algebraically related to and thus uniquely determined by ξ. The case $E = 0$ (the zero mode) must be investigated separately and is given by the classical solution (259). Therefore in both models the eigen-functions come in pairs (except for the zero mode)

$$\psi_\pm = \begin{bmatrix} \xi \\ \pm\rho \end{bmatrix} \text{ with } \omega_\pm = \pm\frac{m}{l}\sqrt{E_{(l)}}$$

For periodic boundary condition it is necessary to change the eigen-basis and work with "parity" eigen-functions

$$\xi^{g,u} = \frac{1}{2}(\xi \pm \xi^*) \ , \quad \rho^{g,u} = (\frac{-i}{\sqrt{E_{(l)}}} A_l \xi^{g,u}) \text{ iff } E \neq 0$$

which form the pairs of solutions

$$\begin{aligned}
u_\pm &= \begin{bmatrix} \xi^g \\ \pm\rho^g \end{bmatrix} \text{ with } \omega_\pm = \pm\tfrac{m}{l}\sqrt{E_{(l)}} \\
\phi_\pm &= \begin{bmatrix} \xi^u \\ \pm\rho^u \end{bmatrix} \text{ with } \omega_\pm = \pm\tfrac{m}{l}\sqrt{E_{(l)}}
\end{aligned}$$

The explicit expressions for $\psi_\pm$ and $u_\pm, \phi_\pm$ are given in the appendix and will be needed to determine the scattering phases. The discrete states fall off fast enough so that they fit in all considered spin structures.

Boundary conditions and symmetry principle

The influence of the kink background on fermionic fluctuation $V'(\phi_K)$ is now in both cases an antisymmetric function (274) and lives therefore on a line bundle with the topology of a

Möbius-strip (like the kink itself). Thus one obtains with an ansatz for ψ for a surrounding of the compactified dimension

$$\begin{aligned} x \to x + L: \quad V'(\phi_K) &\to -V'(\phi_K) \\ \psi &\to A\Gamma\psi \end{aligned}$$

For the different spin structures the Lagrangian, when transported around the compactified dimension, gets the additional contributions

$$P/AP : \Gamma = 1 \qquad \delta\mathcal{L} = V'(\phi_K)\bar{\psi}\psi \tag{279}$$

$$TP/TAP : \Gamma = \gamma_\star \qquad \delta\mathcal{L} = 0 \tag{280}$$

So for the TP/TAP - spin structures one does not pick up a BC-contribution to the action integral. This reflects the residual chiral symmetry of the quadratic part of the expanded action.

5.3.5 Sine Gordon

First we consider the SG- model with the TAP- spin structure

$$\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}(-L/2) = \begin{bmatrix} \psi_2 \\ \psi_1 \end{bmatrix}(L/2) \tag{281}$$

(because of the extra minus due to our metric convention). But as we will see the TP- spin structure is automatically also treated by TPA. By setting up the BC for the modes $\psi_\pm$ the whole field, expanded according to this modes, automatically fulfills the BC.

For the ψ_+ modes (see appendix) one gets for the two components in (281)

$$-ie^{i[k\tilde{L}+\theta^-]} = 1 \quad \Rightarrow \quad k^+\tilde{L} + \theta^- = 2n\pi + \frac{\pi}{2} \tag{282}$$

$$ie^{i[k\tilde{L}+\theta^+]} = 1 \quad \Rightarrow \quad k^+\tilde{L} + \theta^+ = 2n'\pi - \frac{\pi}{2} \tag{283}$$

where $\theta^\pm = \arg(\pm 1 - ik)$ are the arguments of the asymptotic $\xi_+(z \to \pm\infty)$. One can also absorb the factors $\pm i$ and the addend $\pm\frac{\pi}{2}$ in (283) in the angles $\theta^\pm$, but this is pure convention. The two quantization conditions (283) are consistent if $\theta^+ + \frac{\pi}{2} = -(\theta^- + \frac{\pi}{2}) + 2\pi m$ in the considered momentum regime.

The analogous expressions for the ψ_- modes are given by

$$ie^{i[k\tilde{L}+\theta^-]} = 1 \quad \Rightarrow \quad k^-\tilde{L} + \theta^- = 2n\pi - \frac{\pi}{2} \tag{284}$$

$$-ie^{i[k\tilde{L}+\theta^+]} = 1 \quad \Rightarrow \quad k^-\tilde{L} + \theta^+ = 2n'\pi + \frac{\pi}{2} \tag{285}$$

The two quantization conditions are consistent if $\theta^+ - \frac{\pi}{2} = -(\theta^- - \frac{\pi}{2}) + 2\pi m$ for each k.

Quantization phase

We choose θ^+ for our quantization conditions. Its graph is given in fig.13 We consider the continuous phase (a) for which θ^- is also continuous and the consistence equations for the

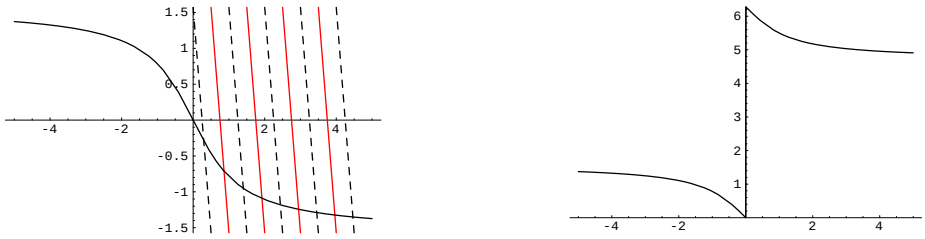


Figure 13: The quantization phase θ^+ for the branch cut positions: (a) $cut = Im_+$, $\arg(z) \in [-3\pi/2, \pi/2]$. Also plotted are the momenta-evens k_n^+ (solid), k_n^- (dashed) with positive solution. (b) $cut = \mathbb{R}_+$, $\arg(z) \in [0, 2\pi]$.

quantization are fulfilled for all momenta simultaneously which makes mode counting much simpler. The quantization condition for the modes $\psi_\pm$ are

$$\tilde{L}k_n^\pm + \theta^+(k_n^\pm) = 2n\pi \mp \frac{\pi}{2}$$

because of the symmetry of the continuous phase the momenta are related as

$$-k_n^+ = k_{-n}^- \tag{286}$$

This is the reason why we did not include the $\frac{\pi}{2}$ into the phase, otherwise in this relation also an index shift occurs. The modes $\psi_\pm$ are related to each other by complex conjugation (see appendix) so that with (286) one has

$$\psi_\pm^*(k_n^\pm) = \psi_\mp(k_{-n}^\mp)$$

So we expand the full fermion field as follows

$$\psi(z,t) = d_0(t)\psi_0(z) + \sum_{k_n^+ \geq 0}^{M_+} \left(a_n(t)\psi_+(k_n^+, z) + a_n^*(t)\psi_+^*(k_n^+, z)\right) \tag{287}$$

$$+ \sum_{k_n^- \geq 0}^{M_-} \left(b_n(t)\psi_-(k_n^-, z) + b_n^*(t)\psi_-^*(k_n^-, z)\right). \tag{288}$$

The first term is the zero mode. Due to the Majorana-condition its Grassmann coefficient is real. Even if we would complexify ψ_0 by a complex normalization factor, d_0^* would depend linearly on d_0. By our choice of the basis and Grassmann valued coefficient functions the Majorana condition $\psi^* = \psi$ is automatically fulfilled for all times.

Inserting (288) in the quadratic action (275) one obtains

$$S_\psi^{(2)} = \int_T dt \left[\frac{i}{2}d_0\dot{d}_0 + \sum_1^{M_+}\left\{\frac{i}{2}(a_n^*\dot{a}_n + a_n\dot{a}_n^*) + \omega(k_n^+)a_n^*a_n\right\}\right.$$
$$\left. + \sum_0^{M_-}\left\{\frac{i}{2}(b_n^*\dot{b}_n + b_n\dot{b}_n^*) + \omega(k_n^-)b_n^*b_n\right\}\right].$$

This is the sum of Grassmann oscillators except for the zero mode which has to be treated by collective coordinates. The path integral measure is up to the zero mode the same as

in the vacuum sector if one takes equal numbers of modes. Neglecting for the moment the subtleties due to the zero mode one can read off of the spectral function the ground state energy contribution due to the fermions:

$$E_F^K = -\frac{\hbar m}{2}\sum_1^{M_+}\sqrt{(k_n^+)^2+1} - \frac{\hbar m}{2}\sum_0^{M_-}\sqrt{(k_n^-)^2+1} + \delta M_F \tag{289}$$

Let us now verify that TP BC give exactly the same result. For TP BC the l.h.s of (281) is multiplied with -1. Therefore only the relations (283) and (285) are interchanged so that ψ_+ has now the momenta from ψ_- in (288) and vice versa so that in the energy (289) only the names $+,-$ are interchanged. To see this directly was the reason why we used the somewhat "lengthy" basis in (288), with another choice, one sum over positive and negative momenta does the same job.

Mode counting

If we apply Levinson's theorem to the (continuous) phase one obtains for the number of bound states

$$\frac{\theta^-(-\infty)-\theta^-(\infty)}{2\pi} = \#_{discrete} = \frac{1}{2} \tag{290}$$

Thus the fermionic zero mode is a half bound state. Note that this has nothing to do with our convection to not include the $\frac{\pi}{2}$ in the definition of the scattering phase in (283) since this constant cancels in the difference in (290). Therefore the continuous spectrum shifts down only by a half mode relative to the vacuum. This cannot be compensated by a discontinuous phase, where an integer number of modes does not have a solution. This information can only read off of the asymptotic values of the scattering phase. For the continuous modes one has

$$\begin{aligned} \text{vacuum}: &\quad \#_V = 2N+1+A \\ \text{kink}: &\quad \#_K = M_+ + M_- + 1 \end{aligned}$$

With the ansatz $M_+ + M_- + 1 = 2N + 1 + A$ one has to subtract in addition the energy of one half high mode (note that the mode energies are negative). The fermionic contribution to the kink mass is therefore (using (271), (289),(263))

$$\begin{aligned} M_F = E_F^K - E_F^V = \; & \frac{\hbar m}{2}\sum_{-(N+A)}^{N}\sqrt{\left(\frac{(2n+A)\pi}{Lm}\right)^2+1} - \frac{\hbar m}{4}\sqrt{\Lambda^2+1} \\ & -\frac{\hbar m}{2}\sum_1^{M_+}\sqrt{(k_n^+)^2+1} - \frac{\hbar m}{2}\sum_0^{M_-}\sqrt{(k_n^-)^2+1} + \delta M_F \end{aligned}$$

For the calculation of the sums we use exactly the same techniques as in the bosonic case (iterative solution for the kink momenta and Euler-MacLaurin). Independently of the splitting of the modes M_- and M_+ and of the vacuum BC ($A = \pm 1$) one obtains for the fermionic kink mass

$$M_F = \frac{\hbar m}{2\pi} + O(\hbar^2) \tag{291}$$

This is the expected and correct result. Note that if we would have counted the zero mode as a full mode the result would be divergent. Of course it is unsatisfactory that we have to, fall back on the Levinson theorem, like Graham and Jaffe [47], and cannot produce this as an implicit result of mode regularization which validate the Levinson theorem. We think that a proper treatment of both zero modes , bosonic and fermionic, with for example collective coordinates and the associated path integral measure will give the desired result and validate the Levinson theorem. This is work in progress.

5.3.6 ϕ^4-model

Next we consider the ϕ^4 model with the TP- spin structure. For the ψ_+ modes (see appendix) one gets for the two components in (281):

$$e^{i[k\tilde{L}+(\alpha^+-\varphi^-)]} = -1 \quad \Rightarrow \quad k^+\tilde{L} + (\alpha^+ - \varphi^-) = (2n+1)\pi \tag{292}$$

$$e^{i[k\tilde{L}+(\varphi^+-\alpha^-)]} = -1 \quad \Rightarrow \quad k^+\tilde{L} + (\varphi^+ - \alpha^-) = (2n'+1)\pi \tag{293}$$

where $\alpha^\pm = \arg(-k \mp i)$ and $\varphi^\pm = \arg(2 - k^2 \mp 3ik)$ are the arguments of the asymptotic state $\rho_+(z \to \pm\infty)$ and $\xi_+(z \to \pm\infty)$. The two quantization conditions (293) are consistent if $(\varphi^+ - \alpha^-) = (\alpha^+ - \varphi^-) + 2\pi m$ in the considered momentum regime. Here again we must be more careful with the choice of the branch cut position as in the bosonic case. Doing this in a consistent way (the same for all angles) one can see that it is not possible to choose the same phase at $-L/2$ for upper and lower components.

The analogous expressions for the ψ_- modes are given by

$$e^{i[k\tilde{L}+(\alpha^+-\varphi^-)]} = 1 \quad \Rightarrow \quad k^-\tilde{L} + (\alpha^+ - \varphi^-) = 2n\pi \tag{294}$$

$$e^{i[k\tilde{L}+(\varphi^+-\alpha^+)]} = 1 \quad \Rightarrow \quad k^-\tilde{L} + (\varphi^+ - \alpha^-) = 2n'\pi \tag{295}$$

The two quantization conditions are consistent if $(\varphi^+ - \alpha^-) = (\alpha^+ - \varphi^-) + 2\pi m$. Again the the TAP BC only interchange the two relations (293) and (295).

Phase shift

There are two branch cut positions for which the consistency of the quantization condition is fulfilled for all momenta simultaneously with $m = 0$, i.e. $\varphi^+ - \alpha^-) = (\alpha^+ - \varphi^-) =: \delta_S(k)$ (S stands for SUSY to distinguish between the bosonic phase δ). This makes mode counting comfortable. Their graphs are given in fig.14 We choose the discontinuous phase. As one can see for both momenta $k^\pm$ the mode $n = 0$ has no solution. For $k \to \infty$ δ_S goes to zero but for $k \to -\infty$ δ_S approaches the value π. It is not possible to choose the branch cut so that the phase is zero for $k = \pm\infty$ as in [31],[32]. This can only be achieved if one, inconsistently, chooses different branch cuts for the angles α and φ. This finite value at $k \to -\infty$ carries the information of the half bound state as we will see. Because of the symmetry of the discontinuous phase and the quantization condition one has for the momenta

$$\tilde{L}k_n^+ + \delta_S(k_n^+) = (2n+1)\pi \tag{296}$$

$$\tilde{L}k_n^- + \delta_S(k_n^-) = 2n\pi \tag{297}$$

$$-k_n^+ = k_{-n}^- \tag{298}$$

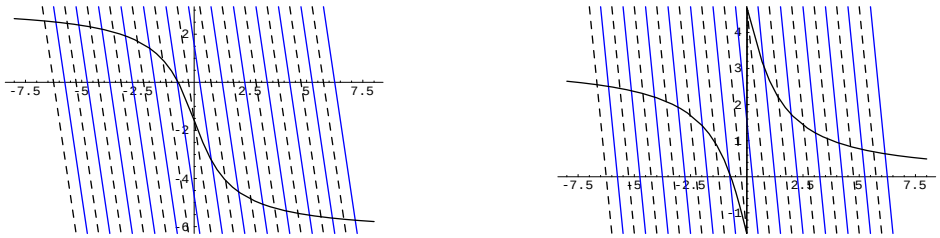


Figure 14: The phase shift δ_S and the momenta evens of the quantization conditions (k_n^+ solid and k_n^-dashed lines) for different branch cut positions: (a) $cut = \mathbb{R}_-$, $\arg(z) \in [-\pi, \pi)$. (b) $cut = \mathbb{R}_+$, $\arg(z) \in [0, 2\pi)$.

Proceeding as in the SG case ($\psi_\pm^*(k) = \psi_\mp(-k)$ and Majorana condition) we get for the full quantum field

$$\psi(z,t) = d_0(t)\psi_0(z) + d_1(t)\psi_1 + d_1^*(t)\psi_1^*(z) + \sum_{-M_-,\neq 0}^{M_+} \left(a_n(t)\psi_+(k_n^+, z) + a_n^*(t)\psi_+^*(k_n^+, z)\right) \tag{299}$$

where we have chosen a more convenient representation than in the SG-case since we already know that TP - BC give the same result. The quadratic action is again the sum of harmonic oscillators and in full analogy to SG (also with respect to the subtleties connected with the zero mode) one obtains for the fermionic kink ground state energy

$$E_F^K = -\frac{\hbar m}{2}\frac{\sqrt{3}}{2} - \frac{\hbar m}{2}\sum_{-M_-,\neq 0}^{M_+}\sqrt{(\frac{k_n^+}{2})^2 + 1} + \delta M_F$$

Mode counting

Applying Levinson's theorem to the continuous phase in fig.14 one obtains

$$\frac{\delta_S(-\infty) - \delta_S(-\infty)}{2\pi} = \#_{discrete} = 1 + \frac{1}{2}$$

Thus again the fermionic zero mode counts as a half mode (bound state), the excited bound state on the other hand counts as a full mode. From (299) the difference between these two modes becomes clear since the excited modes form a pair of complex conjugated pair of degree of freedom (also in the action), in contrast to the zero mode.

The residual calculations are quite analogous to the SG-case. For the fermionic ϕ^4- kink mass one obtains

$$M_F = \hbar m\left(\frac{1}{\pi} - \frac{1}{4\sqrt{3}}\right) + O(\hbar^2)$$

Again this result is obtained independently of the combination of the (allowed) BC for the two sectors.

5.3.7 Anti/periodic spin structures P/AP

Since P/AP- BC induce an additional contribution to the action (279) they are unacceptable for a regularization. But they can be of interest for a nontrivial topology of the universe. As it is shown by [48] a nontrivial topology can violate the CPT symmetry but the derived effects vanish in the large limit of the compactified dimension and it is therefore questionable if such effects will be measurable. The calculation of the kink mass with P/AP on the other hand gives a result that differs from that for TP/TAP BC by a half low-lying mode [37] even in the limit $L \to \infty$. The reason for this is that in the P/AP- spin structure the zero mode is counted as a full mode. For simplicity we consider the SG- model but all considerations and results are analogous for the ϕ^4 model.

Quantization phases. The processing is analogous to the TP/TAP calculation but for P/AP BC one needs the parity eigen states $u_\pm, \phi_\pm$ (see appendix). The parity eigen-states are of the form

$$u_\pm = \begin{bmatrix} \xi^g \\ \pm\rho^g \end{bmatrix} \quad , \quad \phi_\pm = \begin{bmatrix} \xi^u \\ \pm\rho^u \end{bmatrix},$$

where the components are given in the appendix. In $u_\pm$ the upper component ξ^g is an even and the lower component ρ^g a odd function. For $\phi_\pm$ the situation is reversed. Of interest are their asymptotic forms which are given by

$$\begin{aligned} u_\pm: \quad & \xi^g(z \to \pm\infty) = iN_q(\pm \sin qz - q\cos qz) \\ & \rho^g(z \to \pm\infty) = N_q\sqrt{q^2+1}\sin qz \\ \phi_\pm: \quad & \xi^u(z \to \pm\infty) = N_p(\pm\cos pz + p\sin pz) \\ & \rho^u(z \to \pm\infty) = -iN_p\sqrt{p^2+1}\cos pz. \end{aligned}$$

Periodic BC gives no constraints for the even components ξ^g and ρ^u. The odd components must vanish at $\pm L/2$. Thus we get from the asymptotic states:

$$\begin{aligned} \text{periodic } P: \quad & u_\pm(-L/2) = u_\pm(L/2) \;\Rightarrow\; \frac{q\tilde{L}}{2} = n\pi, \\ & \phi_\pm(-L/2) = \phi_\pm(L/2) \;\Rightarrow\; \tan\frac{p\tilde{L}}{2} = -\frac{1}{p}. \end{aligned}$$

The second quantization condition can be resolved as

$$p\tilde{L} + \delta_P(p) = 2n\pi \text{ with } \delta_P(p) = 2\arctan\frac{1}{p}.$$

Thus the eigen-states $u_\pm$ have freely quantized momenta. For antiperiodic BC the odd components are not constrained but the even components must vanish at $\pm L/2$. This gives

$$\begin{aligned} \text{antiperiodic } AP: \quad & u_\pm(-L/2) = -u_\pm(L/2) \;\Rightarrow\; \tan\frac{q\tilde{L}}{2} = q, \\ & \phi_\pm(-L/2) = \phi_\pm(L/2) \;\Rightarrow\; \frac{p\tilde{L}}{2} = \frac{(2n+1)\pi}{2}. \end{aligned}$$

Again we can resolve the nontrivial quantization condition:

$$q\tilde{L} + \delta_{AP}(q) = 2n\pi \text{ with } \delta_{AP}(q) = -2\arctan q.$$

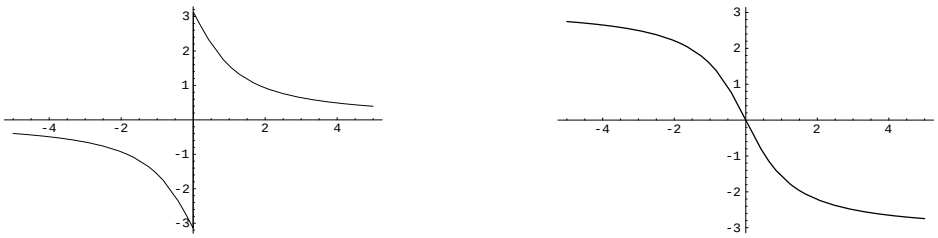


Figure 15: The quantization-phases for (anti)periodic BC. (a) For periodic BC the momenta of $\phi_\pm$ are quantized w.r.t. to a nontrivial scattering phase δ_P. (b) For antiperiodic BC only the momenta of $u_\pm$ are quantized in a nontrivial way by δ_{AP}.

For both cases (P/AP) the scattering (quantization) phase is given in fig.15. The full Majorana-fermion field can be written as

$$\psi(z,t) = \quad d_0\psi_0 + \sum_1^{M_u} \left(a_n(t)u_+(q_n,z) - a_n^*(t)u_+^*(q_n,z)\right) \tag{300}$$

$$+\sum_1^{M_\phi} \left(\alpha_n(t)\phi_+(p_n,z) + \alpha_n^*(t)\phi_+^*(p_n,z)\right). \tag{301}$$

Because of the linear dependence and the symmetry in the quantization conditions only the positive modes occur in the expansion. The modes $n = 0$ do not occur since the associated momentum has no solution or the eigen-mode is the trivial solution. In both cases Levinson's theorem gives one for the number of bound states. But here (301) only half of the continuum modes contribute to the bound state. Therefore it is not completely clear how to really count this mode. Nevertheless only the naive counting of the zero mode as a full mode gives a finite result. This result differs exactly by one half low lying mode from the values calculated above (291) [37]:

$$M_F^K = M_{cl} + \frac{\hbar m}{2\pi} + \frac{\hbar m}{4}$$

This result is obtained independently of the combination of vacuum- and kink- BC, i.e. $(Vac|Kink) = (P, AP|P, AP)$. This is evident since the additional contribution to the action (5.3.2) is in both cases (kink P, AP) the same.

As we have made clear, the origin of this subtlety is tight up with the requirements of a proper treatment of both, bosonic and fermionic zero modes. This suggest a connection to the Atiyah-Singer index theorem for Dirac-operators in a topologically nontrivial background and different topologies of the spin structure.

5.4 Aspects of derivative regularization

In this section we repeat the calculation of [36] for the SUSY-kink mass but in a more general way. The derivative regularization scheme developed in [36] is a proper method for the calculation of the kink mass. We will show now that it is indeed insensitive to the subtleties of mode counting encountered above and that it can therefore be used as a benchmark for other regularization methods which might be inevitable in the calculation of other quantities than the soliton mass.

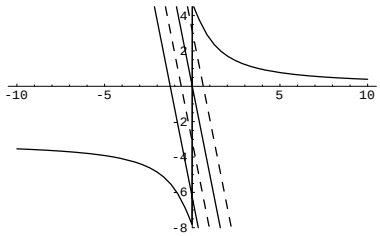


Figure 16: The fermionic scattering phase δ_F of ref. [36] with the skipped modes $n = -1, 0$ (TP/TAP solid/dashed lines). It corresponds to the branch cut at $\mathbb{R}_+$ and has a discontinuity at $k = 0$. The phase jumps by 4π so that two modes do not have a solution and fall out.

Like [36] we shall concentrate on the SUSY-ϕ^4 model. Since the derivative regularization involves the derivative w.r.t. the mass parameter m we re-introduce the physical momenta which are related to the dimensionless ones as $k_{phys} = mk_{dim.-less}$. We also make use of supersymmetry so that the vacuum energies cancel each other. Thus the SUSY- kink mass is given by

$$M_{susy} = (E_B^K - E_B^V) + (E_F^K - E_F^V) = E_B^K + E_F^K$$

For the evaluation of the sums we again use the Euler-MacLaurin formula (318). For ease of comparison with [36] we transform the integration variable n to the physical momentum k according to the quantization condition (TP/TAP-BC)

$$Lk(n) + \delta(k(n)) = (2n + A)\pi$$

where the scattering phase δ is different for bosonic and fermionic fluctuations as will be specified below. Sums are thus evaluated through

$$\sum_{n=\nu}^{N} f(n) = \int_{\nu}^{N} dn f(n) = \int_{\frac{2\nu+A}{L}\pi - \frac{1}{L}\delta(k_\nu)}^{\frac{2N+A}{L}\pi - \frac{1}{L}\delta(k_N)} \frac{dk}{2\pi} \left[L + \delta'(k)\right] f(k) \tag{302}$$

We have omitted the surface terms since they always cancel each other. The integrand $f(k)$ is now, for both fermionic and bosonic fluctuations, given by the derivative of the mode energies w.r.t. the mass m. Including the measure one has:

$$\left[L + \delta'(k)\right] f(k) = \frac{d}{dm}\omega^K(k) = \frac{Lm}{\sqrt{k^2 + m^2}} + \frac{1}{m}\sqrt{k^2 + m^2}\delta'(k) + O(\frac{1}{L}) \tag{303}$$

In principle the scattering phases for bosonic and fermionic fluctuations must be chosen in a consistent way, i.e. one has to select in both cases the same branch cut position. In [36] for the bosonic fluctuations phase (c) of fig.11 where chosen, i.e. the branch cut position $cut = \mathbb{R}_+$. For the fermionic scattering phase [36] used the phase shown in fig.16. For negative k the phase differs by 2π from the phase (b) in fig.14, which is also obtained for a branch cut at $\mathbb{R}_+$. The difference comes only from the fact that in [36] the lower component ρ of the spinor was calculated with the asymptotic values of the upper component ξ in (278). Since the difference is 2π this has no physical meaning and the phase used here also gives consistent quantization conditions like the phase in fig.14.

The choice of the scattering phase in [36] thus suggested that the fermionic zero mode has to be counted as full mode. We show now that the derivative regularization is completely

independent of mode counting arguments and therefore also consistent with the half-counting fermionic zero mode.

For this we leave the magnitude of the discontinuity in the bosonic and fermionic scattering phase arbitrary and assume that the modes $\nu'_{B,F}+1, ...\nu_{B,F}-1$ do not have a solution. Then the derivative of the supersymmetric kink mass reads

$$\frac{dM_{susy}}{dm} = \frac{d(\delta M_{susy})}{dm} + \frac{\hbar}{2}\left(\sum_{N_{B_-},\neq\nu'_B+1,...\nu_B-1}^{N_{B_+}} \frac{d\omega_B^K}{dm} - \sum_{N_{F_-},\neq\nu'_F+1,...\nu_F-1}^{N_{F_+}} \frac{d\omega_F^K}{dm}\right) \tag{304}$$

From (303) one can see that the evaluation of the sums can be split in two parts. The second term in (303) is independent of L and only involves the derivative δ' of the scattering phase which is – for all branch cut positions – the same continuous function. So for this part one can perform the limit $L\to\infty$ in the integral in (302). Together with the counter-term contribution this yields

$$\frac{d(\delta M_{susy})}{dm} + \frac{\hbar}{2m}\int_{-\Lambda}^{\Lambda}\frac{dk}{2\pi}\sqrt{k^2+m^2}\,({\delta_B}' - {\delta_F}') \tag{305}$$

$$= -\frac{\hbar}{2\pi} + \hbar\int_0^{\Lambda}\frac{dk}{2\pi}\frac{1}{\sqrt{k^2+m^2}} + \frac{\hbar}{2m}\int_{-\Lambda}^{\Lambda}\frac{dk}{2\pi}\sqrt{k^2+m^2}\,({\delta_B}' - {\delta_F}') \underset{\Lambda\to\infty}{\to} -\frac{\hbar}{2\pi} \tag{306}$$

This result is independent of the considered numbers of fermionic and bosonic modes as long as the difference between the mode numbers is "small" relative to the highest modes $N_{B,F_\pm}$ so that for $L\to\infty$ this difference vanishes.

Now we have to show that also the L- proportional part of (303) is independent of mode counting arguments. First we investigate the required accuracy for our integration boundaries. In the following we use the abbreviation $\Lambda = \frac{2N\pi}{L}$. Since the integrand is of order $O(\frac{L}{k})$ one has to respect the integration boundaries in (302) up to and including order $O(\frac{1}{L})$. Therefore we can use the following expressions for the scattering phase- contribution to the boundaries (we use the iterative solution for k_n)

$$\frac{1}{L}\delta(k_n) = \frac{1}{L}\delta(\frac{2n+A}{L}\pi) + O(\frac{1}{L^2})$$

For the different boundaries this gives

$$\nu,\nu': \quad \tfrac{1}{L}\delta(\tfrac{2\nu^{(\prime)}+A}{L}\pi) = \tfrac{1}{L}\delta(0_\pm) + O(\tfrac{1}{L^2}) \tag{307}$$

$$N_{B,F_\pm}\approx N: \quad \tfrac{1}{L}\delta(\tfrac{2N+c_{B,F_\pm}+A}{L}\pi) = \tfrac{1}{L}\delta(\Lambda) + O(\tfrac{1}{L^2}) \tag{308}$$

In the second line all numbers $N_{B,F_\pm}$of (304) can differ by an arbitrary amount $c_{B,F_\pm}$ as long as it is not of the order of N, so that $\frac{c_{B,F_\pm}}{L}$ is not of the order Λ. So the residual integral of the difference for bosonic and fermionic contribution is of the form

$$\left[\left(\int_{-\Lambda+\frac{1}{L}\alpha_B}^{\frac{1}{L}\beta_B} + \int_{\frac{1}{L}a_B}^{\Lambda+\frac{1}{L}b_B}\right) - \left(\int_{-\Lambda+\frac{1}{L}\alpha_F}^{\frac{1}{L}\beta_F} + \int_{\frac{1}{L}a_F}^{\Lambda+\frac{1}{L}b_F}\right)\right]\frac{dk}{2\pi}\frac{Lm}{\sqrt{k^2+m^2}} \tag{309}$$

$$\underset{\Lambda\to\infty}{\to} \tfrac{1}{2\pi}(\beta_B - a_B + a_F - \beta_F) \tag{310}$$

The interesting thing is that in the limit $\Lambda\to\infty$ the integration boundary-differences $\frac{1}{L}b_{B,F}$, $\frac{1}{L}\alpha_{B,F}$ do not contribute because of the derivative regularization and so the numbers $c_{B,F_\pm}$

can really be chosen arbitrarily, not only for their phase-contribution (308) but also for the other part of the integration boundary in (302). Without the additional derivative of the mode energies these terms would be of order $O(\Lambda)$ instead $O(\frac{1}{\Lambda})$, and there would be potential linear divergences which cancel each other exactly only by a correct mode counting, especially the half counting of the fermionic zero mode. It is this improvement of convergence by derivative regularization that makes this scheme so robust. But for a complete proof of the independence of mode counting (not only at the high end) we also have to investigate the contribution (310) further. From (307) and (302) we have the following expressions

$$\begin{aligned} a_B &= (2\nu_B + A_B)\pi - \delta_B(0_+) \\ a_F &= (2\nu_F + A_F)\pi - \delta_F(0_+) \\ \beta_B &= (2\nu'_B + A_B)\pi - \delta_B(0_-) \\ \beta_F &= (2\nu'_F + A_F)\pi - \delta_F(0_-) \end{aligned}$$

Independently of the combination of BC for fermionic (A_F) and bosonic (A_B) fluctuations we obtain for (310)

$$\left[(\nu_F - \nu'_F) - \frac{\delta_F(0_+) - \delta_F(0_-)}{2\pi}\right] - \left[(\nu_B - \nu'_B) - \frac{\delta_B(0_+) - \delta_B(0_-)}{2\pi}\right] \tag{311}$$

This expression vanishes independently of branch cut positions and discontinuities of the scattering phase. This can be seen as follows: The jumps of the phases are always integer multiples of 2π, since different (consistent) conventions can only differ by an amount 2π for angles. So the phase terms in (311) is an integer number which is equal to the number $\#_{omit}$ of the modes which have no solution (for the two phases in fig.14 it is 0 for the continuous and 1 for the discontinuous one, and for the phase in fig.16 as used by [36] it is 2). On the other hand the difference of the mode numbers ν, ν', which are the first/last modes with a solution at the discontinuity of the phase, gives one more than the omitted modes $\#_{omit}$. Thus for (311) one obtains

$$(\#^F_{omit} + 1 - \#^F_{omit}) - (\#^B_{omit} + 1 - \#^B_{omit}) = 0$$

So also near zero-momentum the derivative scheme is completely independent of mode counting and therefore insensitive to the subtlety connected with the half-bounded fermionic zero mode. The only contribution to (304) comes from (306), so that

$$\frac{dM_{susy}}{dm} = -\frac{\hbar}{2\pi}$$

Integration w.r.t. m gives therefore the correct result for the SUSY-kink mass

$$M_{susy} = -\frac{\hbar m}{2\pi} + O(\hbar^2)$$

where the integration constant is fixed the normalization $M(m \to 0) = 0$ [36].

5.5 Discussion

We have shown that the derivative regularization scheme is very insensitive to subtleties connected with mode counting. The reason for this is that the potential linear divergence in

the difference of the sums in the mode energies is by differentiation converted into vanishing contributions. Without the derivative this linear divergence is controlled by correct mode counting and the asymptotic values of the scattering phase which is very sensitive on the branch cut position. Also the information about half bounded states is encoded in the asymptotic values of the scattering phase and can as such never be seen in derivative regularization, because only terms with the derivative of the scattering phases contribute in this scheme. The derivative δ' is the same for all branch cut position. As can be seen from (306) it contains the logarithmic divergence which combines and cancels with the counter-term and the supersymmetric part of the fermionic and bosonic kink mass contribution. The non-supersymmetric contribution which despite SUSY gives a correction to the kink mass is completely given by the the derivative of the counter-term contribution.

Because this scheme is so stable against subtleties connected with mode counting it is a good benchmark for regularization schemes that might be required to calculate quantum corrections to other quantities then the kink mass.

6 Conclusion

We have seen that stable non-trivial classical solution play an important role also in the quantized theory. Of special interest are static topological solutions, since they become new particle states in the quantized theory. Their stability is guaranteed by the existence of a topological conservation law. Although we have considered only two-dimensional theories, the conclusions are dimension independent. The existence of a topological conserved current is sufficient for the existence of a Hilbert space sector, which is independent of the usual vacuum sector. Of course it is much more involved to find non-trivial solutions in higher dimensions and as mentioned (Derrick's theorem) it is not possible within a simple scalar field theory but one needs gauge fields for a possible existence of topological non-trivial solution. Another important feature of solitonic solution is that they are non-perturbative results. That is that they are proportional to the inverse coupling constant and thus have an essential singularity in the weak coupling limit. From this is clear that they are not traceable within a standard perturbation theory. Thus they become important in the strongly coupled regime, where standard perturbation theory is not applicable. Especially the possible duality between ordinary quanta of the quantum field theory and bound states of solitons make them very interesting for perturbative quantum theoretical considerations in the strongly coupled regime.

The discussion of the last decade has shown that the quantization procedure in the presence of a non-trivial background like solitons is a highly non-trivial issue. For the renormalization/regularization procedure one has to compare the trivial and non-trivial sector, respectively. The crux is to find a regularization scheme so that the two sectors can be compared in a consistent manner within the regularization procedure, i.e. one needs a consistency relation between the "cutoffs" of the two sectors. We considered mode- energy cutoff- and derivative regularization schemes. We excluded dimensional regularization since the existence of our considered solitonic solutions depends on the dimension. Also we did not consider zeta function regularization, although it is proved to be working perfectly to regularize functional determinants, but it is incompatible with symmetry transformations including fermionic degrees of freedom like supersymmetry [49].

One of the crucial points in *mode regularization* are the boundary conditions which one sets up in the two sectors. As we have shown they can be derived from a very simple symmetry

principle. The power of this symmetry principle, besides its simplicity, is its generality. It is not restricted two dimensions or supersymmetric theories. We were able to show that all combinations of boundary conditions, allowed by this principle, give the unambiguous and correct result for the quantum corrections of the kink masses. To compare the two sectors consistently one has to consider an equal number of modes in the both sectors. This is equivalent to the requirement that the two regularized Hilbert spaces, or path integration domains in functional-language, have the same dimension and are thus isomorphic. There is a profound subtlety connected with the occurrence of zero modes and their counting. Especially in the case of fermions we could resolve this with an additional knowledge gained by the Levinson theorem. This is still an unsatisfactory state within the regularization procedure, but the realization of the correct boundary conditions through our symmetry principle and the need of half counting of the fermionic zero mode is an important step towards the complete resolution of this issue. There seems to be an attractive connection to index theorems of Dirac operators in a non-trivial background in connection with non-trivial spin structures. This of course has to be investigated further and an important point will be the proper treatment of the fermionic zero mode within the framework of collective coordinates and respecting the constraints, which are always present for first order systems like fermions.

For the *energy-momentum cutoff scheme* we have given a heuristic principle to find a relation between the cutoffs in the different sectors. We think that also for the energy-momentum cutoff regularization scheme the consistency between the two sectors is given by the equal dimension (isomorphy) of the different (finite) Hilbert spaces or path integration domains, respectively. It is impossible to set up in a consistent way a common strict cutoff since the presence of the kink changes in a nontrivial way the density of states which determine the dimension of the space in the continuous case. Our heuristic principle of the "equality" of the regularized units is of course only an idea of what really happens. Especially the space dependence of the cutoff is somewhat strange. That the procedure nevertheless works well seems to be a result of the decoupling of the infrared and UV modes. Also the stability against deformations of the space-dependence indicates that the space dependence information is redundant and that a more fundamental principle should not include it. Nevertheless it is questionable weather it is possible to base the energy-momentum cut off regularization on fundamental principles which are completely independent of mode counting arguments, at least for the discrete modes. A more constructive approach, relying on the requirement of isomorphic Hilbert spaces, relates the kink cutoff to the vacuum cut off by setting the integrals over the difference of the densities equal to the number of discrete states, i.e.

$$\int dk \rho_{reg}(k) = \int dk \int_{-\infty}^{\infty} dz \left[\theta(\Lambda_K - |k|)\xi_K^*(k,z)\xi_K(k,z) - \theta(\Lambda_V - |k|)\xi_V^*(k,z)\xi_V(k,z)\right] \stackrel{!}{=} \#_{discrete}.$$

But in spite of the still open problems in energy-momentum cutoff regularization we have showed in an continuum calculation, i.e. *independent* of any boundary conditions, that there is no possibility to set up a common strict cutoff in both sectors.

In addition we have shown that the *derivative regularization scheme*, developed in [36], is completely independent of mode counting arguments. Most subtleties in the regularization procedure are connected with the potentially linearly divergent Casimir-like energy contributions. This is completely circumvented by the derivative regularization. Thus this scheme provides a very robust cross check for other schemes and principles. The disadvantage of the derivative regularization scheme is that it is not applicable for anomaly considerations and

thus, although very comfortable and consistent, one has to use and consistently formulate other regularization schemes in addition.

Summarizing the results of this work we can say that the mode regularization scheme is the one with the most advantages. The invention of the symmetry principle to find the correct boundary conditions, the independence of the combination of the allowed boundary conditions and the realization of the half counting fermionic zero mode are very encouraging results for further investigations of the important issue of non-perturbative quantum field theory

7 Appendix

7.1 Stability equation

The Sine-Gordon and ϕ^4 kink are members of a family of kinks whose zero mode of the stability equation is $\eta_{l,0}(z) = \phi'_K(z) \propto \frac{1}{\cosh^l z}$, where $l = 1$ corresponds to the SG- kink and $l = 2$ to the ϕ^4- kink [41]. The stability equation for this family is given by [40]

$$\mathcal{O}_l \xi = \left(-\partial_z^2 - \frac{l(l+1)}{\cosh^2 z} + l^2 \right) \xi = E\xi$$

and can be solved using supersymmetric methods [40]. The operator $\mathcal{O}_l$ can be factorized into $\mathcal{O}_l = A_l^\dagger A_l$ with

$$A_l = \partial_z + l \tanh z \quad , \quad A_l^\dagger = -\partial_z + l \tanh z$$

The discrete spectrum (bound states) consists of the zero mode

$$\xi_{l,0}(z) = N_l \frac{1}{\cosh^l z} \; , \; E_0 = 0$$

and a set $\{m = 1, \ldots, l-1\}$ of excited states

$$\xi_{l,m}(z) = N_{l,m} A_l^\dagger(z) \ldots A_{l-m+1}^\dagger(z) \left[\frac{1}{\cosh^{l-m} z} \right] \; , \; E_{l,m} = l^2 - (l-m)^2$$

The continuous spectrum has the form

$$\xi_l(q,z) = N_l(q) A_l^\dagger(z) A_{l-1}^\dagger(z) \ldots A_1^\dagger(z) \left[\frac{e^{iqz}}{2\pi} \right] \; , \; E_l(q) = q^2 + l^2 \tag{312}$$

where the factors N are proper normalization constants ($\int dz \xi^*(q,z)\xi(q',z) = \delta(q-q')$). The eigen-functions form a complete set. The mode energies of the quantum fluctuations around the kink are therefore given as

$$\omega_l^K = m\sqrt{E_l}$$

For SG the explicit form of the spectrum is:

$$\omega_0^K = 0 \qquad \xi_0(z) = N_0 \frac{1}{\cosh z} \tag{313}$$

$$\omega^K(q) = m\sqrt{q^2+1} \qquad \xi(q,z) = N(q) \left(\tanh z - iq \right) e^{iqz} \tag{314}$$

For the ϕ^4 model the spectrum is:

$$\omega_0^K = 0 \qquad \xi_0(z) = N_0 \frac{1}{\cosh^2 z} \tag{315}$$

$$\omega_1^K = m\frac{\sqrt{3}}{2} \qquad \xi_1(z) = N_1 \frac{\sinh z}{\cosh^2 z} \tag{316}$$

$$\omega^K(q) = m\sqrt{(\frac{q}{2})^2 + 1} \qquad \xi(q,z) = N(q)\left(3\tanh^2 z - 1 - q^2 - i3q\tanh z\right)e^{iqz} \tag{317}$$

7.2 Fermionic eigen modes

Complex waves $\psi\pm$:

$$\begin{array}{lll} SG-\text{model} \qquad \omega = 0 & \xi_0(z) = N_0\dfrac{1}{\cosh z} \\ & \rho_0 = 0 \\ \omega_\pm(k) = \pm m\sqrt{k^2+1} & \xi((k,z) = N_k(\tanh z - ik)e^{ikz} \\ & \rho(k,z) = -iN_k\sqrt{k^2+1}e^{ikz} \end{array}$$

$$\begin{array}{lll} \phi^4-\text{model} \qquad \omega = 0 & \xi_0(z) = N_0\dfrac{1}{\cosh^2 z} \\ & \rho_0 = 0 \\ \omega_{1,\pm} = \pm m\dfrac{\sqrt{3}}{2} & \xi_1(z) = N_1\dfrac{\sinh z}{\cosh^2 z} \\ & \rho_1(z) = -iN_1\dfrac{1}{\sqrt{3}}\dfrac{1}{\cosh z} \\ \omega_\pm(k) = \pm m\sqrt{(\frac{k}{2})^2+1} & \xi(k,z) = N_k(3\tanh^2 z - 1 - k^2 - i3k\tanh z)e^{ikz} \\ & \rho(k,z) = -iN_k\sqrt{k^2+4}(\tanh z - ik)e^{ikz} \end{array}$$

Parity eigen-functions $u_\pm, \phi_\pm$(continuum states):

$$\begin{array}{llll} SG-\text{model}: & u_\pm: & \xi^g(k,z) = iN_k(\tanh z \sin kz - k\cos kz)\ldots\text{even} \\ & & \rho^g(k,z) = N_k\sqrt{k^2+1}\sin kz\ldots\text{odd} \\ & \phi_\pm: & \xi^u(k,z) = N_k(\tanh z\cos kz + k\sin kz)\ldots\text{odd} \\ & & \rho^u(k,z) = -iN_k\sqrt{k^2+1}\cos kz\ldots\text{even} \end{array}$$

$$\begin{array}{llll} \phi^4-\text{model}: & u_\pm: & \xi^g = N_k\left[(3\tanh^2 z - 1 - k^2)\cos kz + 3k\tanh z\sin kz\right] \\ & & \rho^g = -iN_k\sqrt{k^2+4}(\tanh z\cos kz + k\sin kz) \\ & \phi_\pm: & \xi^u = iN_k\left[(3\tanh^2 z - 1 - k^2)\sin kz - 3k\tanh z\cos kz\right] \\ & & \rho^u = N_k\sqrt{k^2+4}(\tanh z\sin kz - k\cos kz) \end{array}$$

7.3 Euler-MacLaurin formula

The Euler-MacLaurin formula is given by [7]:

$$\sum_{n=\nu}^{N} f(n) = \int_{\nu}^{N} dn f(n) + \frac{1}{2}\left(f(\nu) + f(N)\right) + S_n \tag{318}$$

with

$$S_n := \frac{B_2}{2!} f' + \frac{B_4}{4!} f^{(3)} + \cdots + \frac{B_{2p}}{(2p)!} f^{(2p-1)} \mid_{\nu}^{N} + R_p, \quad p = 2, 3, \ldots,$$

where B_k are the Bernoulli numbers and the rest is

$$R_p = \frac{1}{(2p+1)!} \int_{\nu}^{N} f^{(2p+1)}(x) C_{2p+1}(x) dx.$$

The functions $C_k(x)$ are the modified Bernoulli polynomials.

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